Global Wave Maps on Curved Space Times

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Introduction

Wave maps from a pseudoriemannian manifold of hyperbolic (lorentzian) signature (V, g) into a pseudoriemannian manifold are the generalisation of the usual wave equations for scalar functions on (V, g). They are the counterpart in hyperbolic signature of the harmonic mappings between properly riemannian manifolds.

The wave map equations are an interesting model of geometric origin for the mathematician, in local coordinates they look like a quasilinear quasidiagonal system of second order partial differential equations which satisfy the Christodoulou [17] and Klainerman [18] null condition. They also appear in various areas of physics (cf. Nutku 1974 [6], Misner 1978 [7]).

The first wave maps to be considered in physics were the σ-models, for instance the mapping from the Minkowski spacetime into the three sphere which models the classical dynamics of four meson fields linked by the relation:

$$\sum_{a=1}^{4} \mid f^a \mid^2 = 1 \, .$$

Wave maps play an important role in general relativity, in general integration problem or in the construction of spacetimes with a spatial isometry group. Indeed:

1. The harmonic coordinates, used for a long time in various problems, express that the identity map from (U, g), U domain of a chart of the spacetime, into an open set of a pseudoeuclidean space is a wave map. Wave maps from a spacetime (V, g) into a pseudoriemannian manifold $(V, \hat{e})$, with $\hat{e}$ a given metric on V, gives a global harmonic gauge condition on (V, g).
2. The Einstein, or Einstein-Maxwell, equations for metrics possessing a one parameter spacelike isometry group can be written as a coupled system of a wave map equation from a manifold of dimension three and an elliptic, time dependent, system of partial differential equations on a two dimensional manifold, together with ordinary differential equations for the Teichmuller parameters (Moncrief 1986 [12], YCB and Moncrief 1995 [20]).

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The natural problem for wave maps is the Cauchy problem. It is a nonlinear problem, complicated by the fact that the unknown does not take their values in a vector space, but in a manifold. Gu Chaohao 1980 [9] has proven global existence of smooth wave maps form the 2-dimensional Minkowski spacetime into a complete riemannian manifold by using the Riemann method of characteristics. Ginibre and Velo 1982 [10] have proven a local in time existence theorem for wave maps from a Minkowski spacetime of arbitrary dimensions into the compact riemannian manifolds $O(N)$, $CP(N)$, or $GC(N,p)$ by semigroup methods. They prove global existence on 2-dimensional Minkowski spacetime. These local and global results have been extended to arbitrary regularly hyperbolic sources and complete riemannian targets in YCB 1987 [13], which proves also global existence for small data on $n+1$ dimensional Minkowski spacetime, $n \geq 3$ and odd, due to the null condition property. This last result has been proved to hold for $n = 2$ by YCB and Gu Chaohao 1989 [16], if the target is a symmetric space and for arbitrary n by YCB 1998c [24].

Global existence of weak solutions, without uniqueness, for large data in the case of $2 + 1$ dimensional Minkowski space has been proved by Muller and Struwe 1996 [22]. Counter examples to global existence on $3 + 1$ dimensional Minkowski space have been given by Shatah 1988 [14] and Shatah and Tahvildar-Zadeh 1995 [21].

This article is composed of two parts. In Part A we give a pedagogical introduction to wave maps together with a new proof of the local existence theorem. In Part B we prove a global existence theorem of wave maps in the expanding direction of an expanding universe.

A. General Properties

1 Definitions

Let u be a mapping between two smooth finite dimensional manifolds V and M:

$$u: \quad V \longrightarrow M.$$

Let (x^α), $\alpha = 0, 1, \ldots, n$, be local coordinates in an open set ω of the source manifold V supposed to be of dimension $n+1$. Suppose ω sufficiently small for the mapping u to take its value in a coordinate chart (y^A), $A = 1, \ldots, d$ of the target manifold M supposed to be of dimension d. The mapping u is then represented in ω by d functions u^A of the $n+1$ variables x^α

$$(x^\alpha) \quad \mapsto \quad y^A = u^A(x^\alpha).$$

The mapping u is said to be differentiable at $x \in \omega \subset V$ if the functions u^A are differentiable. The notion is coordinate independent if V and M are differentiable.

The gradient $\partial u(x)$ of the mapping u at x is an element of the tensor product of the cotangent space to V at x by the tangent space to M at $u(x)$:

$$\partial u(x) \in T_x^* V \otimes T_{u(x)} M.$$

The gradient itself, ∂u, is a section of the vector bundle E with base V and fiber $E_x \equiv T_x^* V \otimes T_{u(x)} M$ at x.

We now suppose that the manifolds V and M are endowed with pseudoriemannian metrics denoted respectively by g and h. We endow the vector bundle E with a connexion whose coefficients acting in $T_x^* V$ are the coefficients of the riemannian connexion at x of the metric g while the coefficients acting in $T_{u(x)} M$ are the pull back by u of the connexion coefficients of the riemannian connexion at $u(x)$ of the metric h, we denote by ∇ the corresponding covariant differential. If f is an arbitrary section of E represented in a small enough open set ω of V by the $(n+1) \times d$ differentiable functions f_α^A of the $n+1$ coordinates x, then its covariant differential is represented in ω by the $(n+1)^2 \times d$ functions

$$\nabla_\alpha f_\beta^A(x) \equiv \partial_\alpha f_\beta^A(x) - \Gamma_{\alpha\beta}^\mu(x) f_\mu^A(x) + \partial_\alpha u^B(x) \Gamma_{BC}^A(u(x)) f_\beta^C(x),$$

where $\Gamma_{\alpha\beta}^\mu$ and Γ_{BC}^A denote respectively the components of the riemannian connections of g and h.

The covariant differential of a section f of E is a section of $T_* V \otimes \mathrm{E}$, also a vector bundle over V.

Analogous formulas using the Leibniz rule for the derivation of tensor products give the covariant derivatives in local coordinates of sections of bundles over V with fiber $\otimes\, {}^p T_x^* \mathrm{V} \otimes {}^q \mathrm{T}_{u(x)} M$. In particular:

1. The covariant differential $\nabla \mathrm{g}$ of the metric g, section of $\otimes^2 T^* V$, is zero by the definition of its riemannian connection. The field $h(u)$ defined by u and the metric h, section of the vector bundle over V with fiber $\otimes^2 T_{u(x)}$ at x, has also a zero covariant derivative $\nabla \mathrm{h}$, pull back by u of the riemannian covariant derivative of h.
2. Commutation of covariant derivatives gives the following useful generalisation of the Ricci identity

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) f_\lambda^A = R_{\alpha\beta\lambda}{}^\mu(x) f_\mu^A(x) + \partial_\alpha u^C \partial_\beta u^B R_{CB}{}^A{}_D f_\mu^D.$$

2 Wave Maps. Cauchy Problem

From now on we will suppose that the source (V, g) is lorentzian, i.e. that the metric g is of hyperbolic signature, which we will take to be $(-, +, ..., +)$.

The following definition generalizes to mappings into a pseudoriemannian manifold the classical definition of a scalar valued wave equation on a lorentzian manifold.

Definition. A mapping u: $(V,g) \to (M,h)$ is called a *wave map* if the trace with respect to g of its second covariant derivative vanishes, i.e. if it satisfies the following second order partial differential equation, taking its values in TM:

$$g.\nabla^2 u = 0.$$

In local coordinates on V and M this equation is:

$$g^{\alpha\beta}\nabla_\alpha \partial_\beta u^A \equiv g^{\alpha\beta}(\partial^2_{\alpha\beta} u^A - \Gamma^\lambda_{\alpha\beta}\partial_\lambda u^A + \Gamma^A_{BC}(u)\partial_\alpha u^B \partial_\beta u^C = 0.$$

The wave map equation reads thus in local coordinates as a semilinear quasidiagonal system of second order partial differential equations for d scalar functions u^A. The diagonal principal term is just the usual wave operator of the metric g; the nonlinear terms are a quadratic form in ∂u, with coefficients functions of u.

The wave map equation is invariant under isometries of (V,g) and (M,h): let u be a wave map from (V,g) into (M,h), let f and F be diffeomorphisms of V and M respectively, then $F \circ u \circ f$ is a wave map from $(f^{-1}(V), f_* g)$ into $(F(M), dFh)$.

Throughout this paper we stipulate that the manifold V is then of the type $S \times R$, with each submanifold $S_t \equiv S \times \{t\}$ space like. We denote by (x,t) a point of V.

Remark. If the source (V,g) is globally hyperbolic, i.e. the set of timelike paths joining two points is compact in the set of paths (Leray 1953 [1]), then it is isometric to a product $S \times R$ with each submanifold $S_t \equiv S \times \{t\}$ spacelike and a Cauchy surface, i.e. such that each timelike or null path without end point cuts S_t once (Geroch 1970 [4]).

The first natural problem to solve for a wave map is the *Cauchy problem*, i.e. the construction of a wave map taking together with its first derivative given values on a spacelike submanifold of V for instance S_0. The Cauchy data are a mapping φ from S into M and a section ψ of the vector bundle with base S and fiber $T_{\varphi(x)}$ over x, namely:

$$u(0,x) = \varphi(x) \in M, \quad \partial_t u(0,x) = \psi(x) \in T_{\varphi(x)} M.$$

The results known for Leray hyperbolic systems cannot be used trivially when the target M is not a vector space. However, the standard local in time existence and uniqueness results known for scalar-valued systems can be used to solve the local in time problem for wave maps by glueing local in space results (cf CB 1998a [23]). This local in time existence can also be deduced from those known from scalar valued systems by first embedding the target (M,h) into a pseudoriemannian manifold (Q,q) with Q diffeomorphic to R^n. We give here a variant of the obtention of a system of R^N valued partial differential equations equivalent, modulo hypothesis on the Cauchy data, to the wave map equation.

Lemma 1. Let u: $V \to M$ and i: $M \to Q$ be arbitrary smooth maps between pseudoriemannian manifolds (V,g), (M,h), (Q,q). Set $U \equiv i \circ u$, map from

(V, g) into (Q, q). Denote by ∇ the covariant derivative corresponding to the map on which it acts,then the following identity holds:

$$\nabla \partial U \equiv \partial i . \nabla \partial u + \nabla \partial i . (\partial u \otimes \partial u),$$

that is, if (x^α), (x^A) and (x^a) are respectively local coordinates on V, M and Q while ∇ is the covariant derivative either for the maps u: $(V, g) \to (M, h)$, or i: $(M, h) \to (Q, q)$ or U: $(V, g) \to (Q, q)$,

$$\nabla_\alpha \partial_\beta U^a \equiv \partial_A i^a \nabla_\alpha \partial_\beta u^A + \partial_\alpha u^A \partial_\beta u^B \nabla_A \partial_B i^a.$$

Proof. By the definition of the covariant derivative we have

$$\nabla_\alpha \partial_\beta U^a \equiv \partial^2_{\alpha\beta} U^a - \Gamma^\lambda_{\alpha\beta} \partial_\lambda U^a + \Gamma^a_{bc} \partial_\alpha U^b \partial_\beta U^c,$$

where Γ^a_{bc} are the coefficients of the riemannian connexion of (Q, q),

By the law of the derivation of a composition map we find

$$\partial_\alpha U^a \equiv \partial_\alpha (i \circ u)^a \equiv \partial_A i^a \partial_\alpha u^A,$$

$$\partial^2_{\alpha\beta} U^a \equiv \partial_A i^a \partial^2_{\alpha\beta} u^A + \partial^2_{AB} i^a \partial_\alpha u^A \partial_\beta u^B.$$

The given formula results from these expressions after adding and substracting the term $\partial_A i^a \Gamma^A_{BC} \partial_\alpha u^B \partial_\alpha u^C$ (up to names of summation indices). We obtain as announced:

$$\nabla_\alpha \partial_\beta U^a \equiv \partial_A i^a (\partial^2_{\alpha\beta} u^A - \Gamma^\lambda_{\alpha\beta} \partial_\lambda u^A + \Gamma^A_{BC} \partial_\alpha u^B \partial_\beta u^C)$$
$$+ (\partial^2_{AB} i^a - \Gamma^C_{AB} \partial_C i^A + \Gamma^a_{bc} \partial_A i^b \partial_B i^c) \partial_\alpha u^A \partial_\beta u^B. \quad (1)$$

Lemma 2. Suppose (M, h) is isometrically embedded in (Q, q), i.e. $h \equiv i_* q$, then $\nabla \partial i \in \otimes^2 T_* M \otimes TQ$ is the pull back on M of the second fundamental form K of $i(M)$ as submanifold of Q, it takes its values at a point $y \in i(M)$ in the subspace of $T_y Q$ orthogonal to $T_y i(M)$. We have in arbitrary coordinates on M and Q:

$$\nabla_A \partial_B i^a \equiv \partial_A i^b \partial_B i^c K^a_{cb}.$$

Proof. It is a classical result (cf. for instance [15, V 2, p 280]); it can be proved and explained as follows in adapted local coordinates of M and Q. Let (y^A), $A = 1, ..., d$ be local coordinates in the neighbourhood of a point $y_0 \in M$. We choose in a neighbourhood in Q of the point $i(y_0)$ local coordinates (z^a), $a = 1, ..., D$, such that the embedding i is represented in this neighbourhood by:

$$i^a(y) = y^a \quad \text{if} \quad a = 1, ..., d \quad \text{and} \quad i^a(y) = 0 \quad \text{if} \quad a = d+1, ..., D.$$

We choose a moving frame with d axes such that $\theta^a = dy^a$, $a = 1, ..., d$, while the other $D - d$ axes are orthogonal to these ones and between themselves. In the neighbourhood considered the metric q of Q is then i:

$$q = \sum_{a,b=1}^{d} q_{ab} dy^a dy^b + \sum_{a=d+1}^{D} (\theta^a)^2,$$

The gradient of the mapping i: $(M,h) \to (Q,q)$ in the chosen coordinates and frame is:

$$\partial_A i^a = \delta^a_A, \quad A = 1, ..., d \quad ; \quad a = 1, ..., D.$$

Denote by Q^a_{bc} the connection coefficients of the metric q in the considered coframe, the covariant derivative of the gradient of a mapping i: $(M,h) \to (Q,q)$ is:

$$\nabla_B \partial_A i^a \equiv \partial^2_{BA} i^a - \Gamma^C_{BA} \partial_C i^a + Q^a_{bc} \partial_B i^b \partial_A i^c$$

which gives here:

$$\nabla_B \partial_A i^a = -\Gamma^a_{BA} + Q^a_{BA}, \quad \text{if} \quad a = 1, ..., d;$$
$$\nabla_B \partial_A i^a = Q^a_{BA} \quad \text{if} \quad a = d+1, ..., D.$$

If i is an isometric embedding we have on M that $q_{ab} = h_{ab}$, $a, b = 1, ..., d$. We have then on $i(M)$ identified with M:

$$\Gamma^a_{bc} = Q^a_{bc}, \quad a, b, c = 1, ..., d,$$

while Q^a_{BA}, $B, A = 1, ..., d$; $a = d+1, ..., D$ are the components of the pull back by i of the second fundamental form of $i(M)$ as submanifold on M, equal in the chosen coordinates' frame to the components K^a_{bc} of that form in the chosen frame orthogonal to the tangent space to $i(M)$.
Remark. Denote by $\nu^{(a)}$, $a = d+1, ..., D$, the unit mutually orthogonal vectors orthogonal to i(M). In the chosen coordinates and frame the components of $\nu^{(a)}$ are

$$\nu^{(a)}_c = \delta^a_c \text{if} \quad a, c = d+1, \ldots, D, \quad \nu^{(a)}_c = 0 \text{ifc} = 1, ..., d.$$

We find therefore in this frame

$$\nabla_b \nu^{(a)}_c = -Q^a_{bc}, \quad b, c = 1, \ldots, d; \quad a = d+1, \ldots, D.$$

which gives the usual tensorial form for the components of the second fundamental form of $i(M)$ as an element of $\otimes^2 T_* i(M) \otimes TQ$.
Lemma 3. If the mapping u: $(V,g) \to (M,h)$ is a wave map and if the mapping i: $(M,h) \to (Q,q)$ is an isometric embedding then the mapping $U \equiv i \circ u$: $(V,g) \to (Q,q)$ satisfies in the considered local coordinates the following semilinear second order equation:

$$g^{\alpha\beta}\{\nabla_\alpha \partial_\beta U^a - \partial_\alpha U^c \partial_\beta U^b K^a_{bc}(U)\} = 0.$$

Proof. The proof results from lemmas 1 and 2 together with the fact that $\partial_\alpha U^a \equiv \partial_A i^a \partial_\alpha u^A$.

Suppose that the manifold (M,h) is properly riemannian and has a non-zero injectivity radius. Embed it isometrically in a riemannian space (Q,q)

such that $i(M)$ admits a tubular neighbourhood Ω in Q (geodesics orthogonal to $i(M)$ have a length bounded away from zero in this neighbourhood). The subset $\Omega \subset Q$ can be covered by domains of local coordinates of the previously considered type with $K(U)$ depending smoothly on U in Ω.

The system satisfied by U: $(V, g) \to (\Omega, q)$ is invariant by change of coordinates on M and $\Omega \subset Q$. We can write it intrinsically under the form, with $K(U)$ defined when $U \in \Omega$:

$$\{\nabla^2 U - K(U).(\partial U \otimes \partial U)\} = 0,$$

where the first dot is a contraction in g and the second dot a contraction in q.

Choose Q diffeomorphic to R^N, as it is always possible (Whitney theorem), then there exists global coordinates z^I on Q. In these coordinates the equation satisfied by the mapping U: $(V, g) \to (Q, q)$ reads as a system of second order semilinear system of partial differential equations for a set of scalar functions U^I, defined if $U \equiv \{U^I\} \in \Omega$.

If (M, h) is properly riemannian it is always possible (Nash theorem) to embed it isometrically in a euclidean space (R^N, e). If M is compact then $i(M)$ always admit a tubular neighbourhood Ω.

Remark. If q is a flat metric, the operator $g.\nabla^2 U$ reads as a linear operator, the usual wave operator on (V, g) for a set of scalar functions, when the coordinates z^I are the cartesian ones, the nonlinearities are concentrated in the term with coefficient K.

3 Local Existence Theorem. Global Problem

We will use the classical local existence theorem for Leray hyperbolic system applied to the system we have obtained for U by embedding (M, h) for instance in a euclidean space.

We first recall some definitions. We denote by greek letters spacetime indices while tensors on S are indexed with latin letters. A metric $\mathbf{g}$ on $V \equiv S \times R$ is written in boldface, a t dependent metric on S is denoted g_t or (g_{ij}). We write as usual the spacetime metric $\mathbf{g}$ in a moving frame with time axis at the point (x, t) orthogonal to S_t under the form:

$$\mathbf{g} = -N^2 dt^2 + g_{ij}\theta^i\theta^j, \quad \text{with} \quad \theta^i \equiv dx^i + \beta^i dt.$$

The function N, called lapse, is strictly positive; the vector β is called the shift; the induced metric on each S_t, $g_t \equiv g_{ij}dx^i dx^j$, is properly riemannian.

Definition 1. Let $I \equiv [t_0, L)$ be an interval of R. The hyperbolic metric $\mathbf{g}$ on $V \equiv S \times I$ is said to be regularly hyperbolic if:

(i) There exist positive and continuous functions of t, B_1 and B_2, such that for each $t \in I$ it holds on S that

$$0 < B_1 \leq N \leq B_2.$$

(ii) The metrics g_t, $t \in I$, induced by $\mathbf{g}$ on S_t, are equivalent to a given riemannian metric s on S,, that is, there exist positive and continuous functions of $t \in I$, A_1 and A_2, such that for any vector field ξ on S and $t \in I$ it holds on S that

$$A_1 s(\xi, \xi) \leq g_t(\xi, \xi) \leq A_2 s(\xi, \xi).$$

We suppose that the metric s has a non zero injectivity radius, hence is complete. The same property is then enjoyed by each g_t and the manifold $(V, \mathbf{g})$ is globally hyperbolic (CB 1967 [3]).

We now define functional spaces of tensor fields on S.

Definition 2. The Sobolev space W_s^p of tensors of some given type on S is the completion of the space of such tensors in C_0^∞ (i.e. infinitely differentiable with compact support on S) in the following norm:

$$\| f \|_{W_s^p} \equiv \{\sum_{k=0}^{p} \int_S | D^k f |^p \mu_s\}^{1/p},$$

where D denotes the covariant derivative, $| \ |$ the pointwise norm and μ_e the volume element in the metric e. We set $W_s^2 = H_s$.

With the given definition of the spaces W_s^p and the hypothesis that e has a nonzero injectivity radius the usual imbedding and multiplication properties of Sobolev spaces on R^n hold.

Our spaces W_s^p coincide with spaces of tensor fields whose generalised covariant derivatives in the metric e of order less or equal to p are in $L^p(\mu_s)$ if in addition to previous hypothesis we suppose that the curvature of the metric s is uniformly bounded as well as its derivatives of relevant order (cf. Aubin 1982 [11]).

Remark. For the local existence theorem the hypothesis that s has a nonzero injectivity radius can be replaced by its Sobolev regularity, i.e. by the hypothesis that the Sobolev embedding and multiplication properties hold: it is the case when (S, s) is a bounded open set of R^n enjoying the cone property (cf. for instance C.B-D.M [15, V 2, p 379]).

We now define functional spaces for tensor fields on V, noting first that a tensor of order P on V can be decomposed into a finite number of t-dependent tensors of order $\leq P$ on S. We say that the restriction to some given t of a tensor f on V belongs to a given functional space on S if it is so for each tensor of the above decomposition.

For simplicity of writing we take in this section the initial submanifold to be $t_0 = 0$.

Definition 3. We denote by $E_s^p(T)$ the Banach space of tensor fields on $V_T \equiv S \times [0, T]$ defined by

$$E_s^p(T) \equiv \mathrm{C}^k([0, T], W_{s-k}^p), \quad 0 \leq \mathrm{k} \leq \mathrm{s}.$$

We denote by E^p_s a space of tensors on V which are in $E^p_s(T)$ for any finite T. We set $E_s \equiv E^2_s$

Embedding and multiplication properties of the spaces $E^p_s(T)$ are an immediate consequence of these properties for the spaces W^p_s.

Theorem. Let $(V \equiv S \times I, \mathbf{g})$ be a regularly hyperbolic manifold with $[0,T] \subset I$. Let (M,h) be a smooth complete riemannian manifold embedded by i in an euclidean space R^N with cartesian coordinates z^I. Suppose that $D\mathbf{g}, \partial_t \mathbf{g} \in E_{s-1}(T)$.

Let φ, ψ be Cauchy data on S for a wave map $(V,\mathbf{g}) \to (M,h)$. Suppose that the corresponding set of functions $\Phi^I = (i \circ \varphi)^I$ and $\Psi^I = \partial i^I \psi$ on S are such that

$$\Phi^I \in H_s \text{and} \Psi^I \in H_{s-1}$$

Then if $s \geq \frac{n}{2} + 1$ there exists $\ell > 0$ and a wave map u taking the given data, and such that $U^I \equiv (i \circ u)^I \in E_s(\ell) \cap \Omega$.

The interval ℓ of existence for any s is equal to the interval corresponding to $s = s_0$, smallest integer greater than $\frac{n}{2} + 1$.

The solution is unique and depends continuously on the data. A solution with $U \in E_{s_0}(\ell)$, can be approximated by solutions with U in $E_s(\ell)$.

In the case $n = 2$ or 3 the result holds for $s_0 = 2$.

Proof. The existence and properties of $U \equiv (U^I)$ is classical (Leray theory, as completed by Dionne 1962 [2], YCB 1971 [5], YCB-Christodoulou-Francaviglia 1979 [8], one uses the fact that E_{s-1} is an algebra when $s - 1 > \frac{n}{2}$. The extension to $s = 2$ in the case $n = 2$ or 3 has been proved on Minkowski spacetime by Klainerman and Machedon [18], on curved spacetimes by Sogge 1993 [19](Fourier method) and C-B 1998a [23](energy estimates).

To show that U defines a wave map u taking the given Cauchy data we return to our adapted coordinates y^a in $\Omega \subset Q \equiv R^N$. If there exists a mapping u: $V \to M$ such that $U = i \circ u$, i.e. if U takes its values in $i(M)$, then we have the identity

$$g^{\alpha\beta}\{\nabla_\alpha \partial_\beta U^a - \partial_\alpha u^A \partial_\beta u^B \nabla_A \partial_B i^a\} \equiv \partial_A i^a g^{\alpha\beta} \nabla_\alpha \partial_\beta u^A.$$

The mapping U: $V \to i(M) \subset \Omega \subset Q$ annuls the left hand side, the right hand side is then also zero and u is a wave map taking the given Cauchy data. We thus have only to prove that U takes its values in $i(M)$, i.e. that $U^a = 0$ for $a = d+1, \ldots, D$. The equation satisfied by U reads:

$$g^{\alpha\beta}\{\partial^2_{\alpha\beta} U^a - \Gamma^\lambda_{\alpha\beta} \partial_\lambda U^a + \partial_\alpha U^b \partial_\beta U^c (Q^a_{bc} \text{ - } K^a_{bc})$$

with $K^a_{bc} = Q^a_{bc}$ if $a = d+1, \ldots, D$ (note also that $K^a_{bc} = 0$ for $a = 1, \ldots, d$), hence the $D - d$ functions U^a, $a = d+1, \ldots, D$, satisfy a linear homogeneous system, with zero Cauchy data by hypothesis. This system is only local, as are the coordinates y^a, however it is not difficult to deduce from it that U takes its values in $i(M)$ by using a partition of unity and the finite propagation speed of solutions of the wave equation.

Corollary. The theorem can be extended to local spaces, i.e. by replacing the spaces W^p_s on S by spaces of functions which are in W^p_s in each open relatively compact set $\omega_{(i)}$ of some locally finite covering of S, with uniformly bounded $W^p_s(\omega_{(i)})$ norms (cf. C-B 1998a).

Remark. It is possible to prove an analogous theorem with variants on the hypothesis on the metric $\mathbf{g}$. For instance less time regularity or (and) replacement of the spaces H_s on S by spaces W^p_s. One obtains eventually less time regularity of the solution.

The hypothesis made on the metric imply in all cases that $D\mathbf{g}$ is uniformly bounded on V_T. They do not necessarily imply that it is lipshitzian: the geodesics between two nearby points may not be unique.

Global existence lemma. Let $(V \equiv S\times[0,\infty),\mathbf{g})$ be a regularly hyperbolic manifold with $D\mathbf{g}$, $\partial_t\mathbf{g} \in E_s$, $s \geq s_0$. The wave map u with Cauchy data φ, ψ such that $(\Phi,\Psi) \in H_s \times H_{s-1}$. Then u exists globally on V if the norms $\| U(t,.) \|_{H_s}$ and $\| U(t,.) \|_{H_{s-1}}$ do not blow up in a finite time, i.e. are bounded by functions of t continuous on the interval $I \equiv [0,\infty)$.

Proof. It is a standard consequence of the local existence theorem, with the continuous dependence of the interval of existence on the $H_{s_0} \times H_{s_0-1}$ norm of the data.

In the next sections we will endeavour to estimate the involved $H_s \times H_{s-1}$ norms

4 First Energy Estimate

To study global problems for wave maps one must use their special geometric properties, as for other fundamental equations of physics.

The first quantity of physical significance is the energy of the map. In contradistinction with the case, where the source is riemannian, the energy of the map is not the spacetime Dirichlet integral (which is not a positive quantity in the lorentzian case) but a space integral analogous to the energy associated with a solution of the wave equation. We introduce it now.

The *stress energy tensor* of a mapping u: $(V,g) \to (M,h)$ is the covariant 2-tensor on V given by:

$$\mathrm{T(u)} \equiv (\mathrm{hou})(\partial\mathrm{u},\partial\mathrm{u}) - \tfrac{1}{2}\, \mathrm{g}\{\mathrm{g}\otimes(\mathrm{hou})\}.\{\partial\mathrm{u}\otimes\partial\mathrm{u}\}$$

that is

$$T_{\alpha\beta} = h_{AB}(u)\partial_\alpha u^A \partial_\beta u^B - \tfrac{1}{2} g_{\alpha\beta} g^{\lambda\mu} h_{AB}(u)\partial_\lambda u^A \partial_\mu u^B$$

which we will usually write:

$$T_{\alpha\beta} \equiv \partial_\alpha u.\partial_\beta \mathrm{u} - \tfrac{1}{2} g_{\alpha\beta}\partial_\lambda u.\partial^\lambda \mathrm{u}.$$

Indices are raised with $\mathbf{g}$, a dot denotes the scalar product in the metric h of the target space.

Lemma 1. The stress energy tensor $T(u)$ of a wave map u has zero divergence.

Proof. The metrics $\mathbf{g}$ and h have zero covariant derivative, therefore

$$\nabla_\alpha T^\alpha_\lambda \equiv \partial_\lambda u.g^{\alpha\beta}\nabla_\alpha\partial_\beta u \equiv h_{AB}(u)\partial_\lambda u^A g^{\alpha\beta}\nabla_\alpha\partial_\beta u^B = 0$$

if u is a wave map.

Corollary. The stress energy tensor of the mapping $U \equiv i o u$: $(V, \mathbf{g}) \to (R^N, q)$, i an isometric embedding of (M, h) into (R^N, q) has zero divergence if u is a wave map.

Proof. If (M, h) is isometrically embedded by i in (R^N, q) then the stress energy tensors of u and $U \equiv i \circ u$ are the same tensors on V, as can be seen by elementary calculus.

The *energy momentum vector* of the mapping u, equivalently of $U = i o u$, with respect to a vector X on V is the vector $\mathcal{P}(X, u)$ on V given in local coordinates by

$$\mathcal{P}^\alpha \equiv T^\alpha_\beta X^\beta$$

Lemma 2. If X is time like or null, then $\mathcal{P}$(X,u) is time like or null, X and $\mathcal{P}(X, u)$ have opposite time orientation.

Proof. Straightforward, cf. CB 1998a [23].

Lemma 3. The divergence of the energy momentum vector $\mathcal{P}(X, u)$ is given by

$$\nabla_\alpha \mathcal{P}^\alpha = \frac{1}{2}T^{\alpha\beta}(L_X\mathbf{g})_{\alpha\beta}, \qquad (L_X\mathbf{g})_{\alpha\beta} \equiv \nabla_\alpha X_\beta + \nabla_\beta X_\alpha.$$

The energy momentum vector $\mathcal{P}$ has zero divergence if X is a Killing vector of $\mathbf{g}$.

Proof. Straightforward, using the fact that the stress energy tensor has zero divergence. The symmetric 2-tensor $\pi \equiv L_X\mathbf{g}$ is the Lie derivative of the spacetime metric with respect to X.

The energy density of a mapping u at time t with respect to the past oriented timelike or null vector X is the non negative number

$$\epsilon(X, \nu) \equiv \mathcal{P}^{\ \alpha}\nu_\alpha$$

with $\mathcal{P}^{\ \alpha}$ the components of the energy momentum vector $\mathcal{P}(X, u)$ of u with respect to X and ν_α the components of the past oriented unit normal ν to S_t.

The mappings u and $U = i \circ u$ have the same energy density if i is an isometric embedding.

In the coframe $\theta^{\ \alpha}$ we have

$$\nu_i = 0, \quad \nu_0 = N$$

hence

$$\mathcal{P}^\alpha \nu_\alpha = \mathcal{P}^0 N.$$

If the space time metric $\mathbf{g}$ is stationary, i.e. admits a time like Killing vector, it is appropiate to define the energy density with respect to this vector. Otherwise the natural geometric choice is to take for X the past oriented unit normal ν to S_t. The energy momentum vector is then $\mathcal{P}(\nu,\nu)$ and one obtains the usual energy density of u (equivalently of $U \equiv i \circ$ u), denoted $\epsilon(u)$, namely:

$$\mathcal{P}\,^0 N \equiv \epsilon(u) \equiv T^{00}N^2 \equiv \tfrac{1}{2}(\mid N^{-1}\partial_0 u \mid^2_h + \mid Du \mid^2_{g,h})$$

Also, if i is an isometric embedding in a euclidean space (R^N, δ) and $U \equiv i$ou,

$$\epsilon(u) \equiv \tfrac{1}{2}(\mid N^{-1}\partial_0 U \mid^2 + \mid DU \mid^2_g) \equiv \tfrac{1}{2}\delta_{IJ}\{g^{ij}\partial_i U^I \partial_j U^J + N^{-2}\partial_0 U^I \partial_0 U^J\},$$

We have denoted by $\mid\ \mid_{g,h}$ (respectively $\mid\ \mid_g$) the norm both in g and h (respectively in g and δ).

The integral of the energy density of u on S_t is, by definition, the *energy* $e(t,u)$ of u *at time* t. We denote by μ_t the volume element of g_t, we have:

$$e(t,u) \equiv \int_{S_t} \mathcal{P}^0 N \mu_t.$$

We deduce from the hypothesis that $\mathbf{g}$ is uniformly equivalent to the given metric e on S that $|\mathrm{Du}|^2_{g,h} \equiv \mid DU \mid^2_g$ is uniformly equivalent to $\mid DU \mid^2_e \equiv \mid DU \mid^2$. We see that $e(t,u)$ is uniformly equivalent to a sum of norms defined in Sect. 3: there exist positive numbers $C_{\mathbf{g}}$ and $C'_{\mathbf{g}}$ depending only on the bounds on g and N such that:

$$C_{\mathbf{g}} e(t,u) \leq \parallel \partial_0 u(.,t) \parallel_{L^2} + \parallel Du(.,t) \parallel_{L^2} \leq C'_{\mathbf{g}} e(t,u)$$

We denote by K the extrinsic curvature of S imbedded in $(V, \mathbf{g})$. In local coordinates (t, x^i) we have

$$K_{ij} = -\frac{1}{2N}(\partial_t g_{ij} + \nabla_i \beta_j + \nabla_j \beta_i)$$

We will prove the following theorem.

Theorem 1. (energy equality). Let u be a solution of the wave map equation on a manifold $V = \mathrm{S} \times I$ with a C^1 regularly hyperbolic metric $\mathbf{g}$ such that DN and NK are uniformly bounded in g norm on each S_t. Suppose that $u \in C^2(T) \cap E_1(T)$. Then u satisfies for $t \in I \equiv [0,T]$ the fundamental energy inequality:

$$e(t,u) = e(0,u) + \int_0^t \int_{S_\tau} N^{-1}\partial_i N \partial^i u.\partial_0 u + NK^{ij}T_{ij}\}\mu_\tau d\tau$$

Proof. A straightforward computation shows that for $X = \nu$ we have:

$$(L_X\mathbf{g})_{0i} = -\partial_i N, \qquad (L_X\mathbf{g})_{00} = 0, \qquad (L_X\mathbf{g})_{ij} = -2\ \omega\ ^0_{ij} N = K_{ij}$$

The integration of the divergence equation satisfied by $\mathcal{P}$, the value of $L_\nu \mathbf{g}$ and the density of $C_0^\infty(S)$ in H_1 give the theorem.

In cosmological problems it is often convenient to take as time parameter the mean extrinsic curvature of the submanifolds S_t, which characterises the expansion (or contraction) of the universe. We set:

$$\tau \equiv Tr_g K \equiv g^{ij} K_{ij}$$

We will deduce from the energy equality the following corollary.

Corollary. We set

$$P_{ij} \equiv K_{ij} - \frac{1}{n} g_{ij}\tau, \quad \text{with} \quad Tr_g P \equiv g^{ij}P_{ij} = 0.$$

Then

$$e(t,u) = e(0,u) + \int_0^t \int_{S_s} N^{-1}\partial_i N \partial^i u.\partial_0 u + NP^{ij}\partial_i u.\partial_j u\}$$
$$+ N\tau\{(\frac{1}{n} - \frac{1}{2})\{\mid Du \mid^2_{g,h} + \frac{1}{2} \mid N^{-1}\partial_0 u \mid^2_h\}\mu_s ds.$$

Proof. We have:

$$K^{ij}T_{ij} \equiv \{P^{ij} + \frac{1}{n}g^{ij}\tau\}\{\partial_i u.\partial_j u - \frac{g_{ij}}{2}(-N^{-2}\partial_0 u.\partial_0 u + \mid Du\mid^2_{g,h}\}$$

that is

$$K^{ij}T_{ij} \equiv P^{ij}\partial_i u.\partial_j u + \tau\{(\frac{1}{n} - \frac{1}{2})\{\mid Du\mid^2_{g,h} + \frac{1}{2}\mid N^{-1}\partial_0 u\mid^2_h\}$$

Theorem 2. (General energy inequality). Under the hypothesis of Theorem 1 the energy of a wave map satisfies the following inequality:

$$e(t,u) \leq e(0,u)exp\{\int_0^t Sup_{S_\tau}(\mid DN\mid_g + C\mid NK\mid_g)d\tau.$$

with C a positive number depending only on n.

Proof. The integral equality of Theorem 1 together with the inequality satisfied by scalar products imply the following inequality, with C a positive number depending only on n:

$$e(t,u) \leq e(0,u) + \int_0^t \int_{S_\tau}\{\mid DN\mid_g\mid Du\mid_{g,h}\mid N^{-1}\partial_0 u\mid_h + C\mid NK\mid_g(\mid Du\mid^2_{g,h} + \mid N^{-1}\partial_0 u\mid^2_h\}\mu_\tau d\tau$$

hence

$$e(t,u) \leq e(0,u) + \int_0^t Sup_{S_\tau}(\mid DN\mid_g + C\mid NK\mid_g)e_\tau(u)d\tau.$$

This inequality implies the theorem by the Gromwall lemma.

Remark 1. In the case where X is a Killing vector field of **g** and we use it to define the energy density the energy inequality becomes an equality, expressing the conservation of energy of the mapping u. We have chosen here for X the unit normal to S. It is a Killing field if $DN = 0$ and $K = 0$ the corresponding energy $e(t,u)$ is then conserved .

Remark 2. The energy inequality gives only an estimate of ∂U. An estimate of U, as a mapping in R^N, can be obtained from its initial data by the formula

$$U^I(.,t) = U^I(.,0) + \int_0^t \partial_t U^I(.,\tau)d\tau$$

which implies

$$\parallel U(.,t)\parallel_{L^2} \leq \parallel U(.,0)\parallel_{L^2} + t^{1/2}\parallel \partial_t U\parallel_{L^2}.$$

We will return later to the exploitation of the corollary of Theorem 1.

5 Second Energy Estimate

The estimate of the L^2 norm of Du and $\partial_0 u$ on S_t is not sufficient to prove the existence of strong solutions of the wave map eqation even for $n = 1$.

We will now obtain a local in time estimate of the H_1 norms of these quantities by a new method which will be better suited for the cosmological problems. We suppose the shift to be zero, then $\partial_0 \equiv \partial/\partial t$. We denote by $\bar{\nabla}$ the covariant derivative for mappings between the riemannian manifolds $(S,g) \to (M,h)$, acting on sections of vector bundles $\bar{E}^{(p,q)}$ over S with fiber $\otimes T^*_x \otimes^q T_{u(x)}M$, for example:

$$\bar{\nabla}_i\partial_j u^A \equiv \partial^2_{ij}u^A - \Gamma^h_{ij}\partial_h u^A + \Gamma^A_{BC}(u)\partial_i u^B\partial_j u^C.$$

We set (suggestion due to V. Moncrief):

$$e^{(1)}(t,u) \equiv \frac{1}{2}\int_{S_t}\{\bar{\Delta}u.\bar{\Delta}u + \mid \bar{\nabla}u' \mid^2_{g,h}\}\mu_t, \quad \text{with} \quad \text{u'} \equiv N^{-1}\partial_0 u.$$

where $\bar{\Delta}$ is the laplace operator for the metric g and the derivative $\bar{\nabla}$, i.e.:

$$\bar{\Delta}u \equiv g^{ij}\bar{\nabla}_i\partial_j u$$

We denote by D_t the covariant derivative of a mapping from R into time-dependent sections of a vector bundle $\bar{E}$, defined by:

$$D_t\partial_i u^A \equiv \partial_0\partial_i u^A + \Gamma^A_{BC}(u)\partial_0 u^B\partial_i u^C.$$

D_t is a linear operator mapping the space of time-dependent sections of $\bar{E}^{(p)}$ into itself given by the law

$$D_t\bar{\nabla}^p u^A \equiv \partial_0\bar{\nabla}^p u^A + \Gamma^A_{BC}(u)\partial_0 u^B\bar{\nabla}^p u^C.$$

The $\bar{\nabla}$ or D_t derivatives of the mappings from S or R into $\otimes\ ^2TM$ by $x \mapsto h(u(x,t))$ or $t \mapsto h(u(.,t)$ are both zero. The D_t derivative of the metric g_{ij} is equal to $\partial_0 g_{ij} = -2NK_{ij}$. The following commutation relations can be foreseen and checked by straightforward computation:

$$D_t\partial_i u = \bar{\nabla}_i\partial_0 u,$$

$$D_t\bar{\nabla}_i\partial_0 u^A\text{-}\ \bar{\nabla}_i D_t\partial_0 u^A = R^A_{CD\ E}(u)\partial_0 u^C\partial_i u^B\partial_0 u^E,$$

$$D_t\bar{\nabla}_i\partial_j u^A\text{-}\ \bar{\nabla}_i D_t\partial_j u^A = R^A_{CD\ E}(u)\partial_0 u^C\partial_i u^B\partial_j u^E \text{ - } \partial_h u^A\partial_0\Gamma^h_{ij}.$$

We recall the identities (zero shift)

$$\partial_0 g^{ij} = 2NK^{ij}.$$

$$\partial_0\Gamma^h_{ij} \equiv \bar{\nabla}^h(NK_{ij}) - \bar{\nabla}_i(NK^h_j) - \bar{\nabla}_j(NK^h_i).$$

from which we deduce

$$D_t\bar{\nabla}^i\partial_i u = \bar{\nabla}^i D_t\partial_i u + R_{CD\ E}\partial_0 u^C\partial^i u^D\partial_i u^E + 2NK^{ij}\bar{\nabla}_j\partial_i u$$
$$+\{-\bar{\nabla}^h(N\tau) + 2\bar{\nabla}_i(NK^{ih})\}\partial_h u.$$

Before computing the time derivative of $e^{(1)}$ we set:

$$I_0 = \frac{1}{2}\mid\bar{\nabla}u'\mid^2_{g,h}, \quad I_1 \equiv \frac{1}{2}\bar{\Delta}u.\bar{\Delta}u,$$

hence

$$e^{(1)}(t,u) \equiv \int_{S_t}\{I_0 + I_1\}\mu_t.$$

We have

$$\frac{\partial\mu_t}{\partial t} = -N\tau$$

hence

$$\frac{de^{(1)}}{dt} = \int_{S_t}\{\frac{\partial}{\partial t}(I_0 + I_1) - N\tau(I_0 + I_1)\}\mu_t$$

We find, using the definition of D_t, the Leibnitz rule and the property $D_t h = 0$

$$\partial_0 I_1 \equiv D_t I_1 = D_t\bar{\nabla}^i\partial_i u.\bar{\nabla}^j\partial_j u,$$

We have

$$D_t\bar{\nabla}^i\partial_i u = g^{ih}D_t\bar{\nabla}_h\partial_i u + \partial_0 g^{ih}\bar{\nabla}_h\partial_i u$$

with

$$\partial_0 g^{ih} = 2NK^{ih}.$$

Using the commutation formulas and Stokes' formula we obtain:

$$\int_{S_t} D_t I_1 \mu_t = \int_{S_t} \{-D_t \partial_i u.\bar{\nabla}^i \bar{\nabla}^j \partial_j u + \mathrm{F}.\bar{\nabla}^j \partial_j u\} \mu_t$$

with

$$F \equiv R_{CD\ E} \partial_0 u^C \partial^i u^D \partial_i u^E + 2NK^{ij} \bar{\nabla}_j \partial_i u + \{-\bar{\nabla}^h (N\tau) + \mathbf{2}\bar{\nabla}_i (NK^{ih})\} \partial_h u.$$

On the other hand

$$\partial_0 I_0 \equiv D_t I_0 = g^{ij} D_t \bar{\nabla}_i u'.\bar{\nabla}_j u' + \tfrac{1}{2} \partial_0 g^{ij} \bar{\nabla}_i u'.\bar{\nabla}_j u'.$$

therefore:

$$\int_{S_t} D_t I_0 \mu_t = \int_{S_t} \{g^{ij} D_t \bar{\nabla}_i u'.\bar{\nabla}_j u' + NK^{ij} \bar{\nabla}_i u'.\bar{\nabla}_j u'\} \mu_t.$$

We compute the wave map equation $N^{-2}\nabla_0 \partial_0 u^A - g^{ij} \nabla_i \partial_j u^A = 0$ with our definitions. We have, with $\omega^{\alpha}_{\beta\gamma}$ the connection coefficients of $\mathbf{g}$

$$\nabla_0 \partial_0 u^A \equiv \partial_0 \partial_0 u^A - \omega^{\alpha}_{00} \partial_\alpha u^A + \Gamma^A_{BC} \partial_0 u^C \partial_0 u^D$$

which gives:

$$\nabla_0 \partial_0 u^A \equiv N\partial_0 (N^{-1} \partial_0 u^A) - N\partial^i N \partial_i u^A + \Gamma^A_{BC} \partial_0 u^C \partial_0 u^D$$
$$\equiv N\{D_t (N^{-1} \partial_0 u^A - \partial^i N \partial_i u^A\}.$$

On the other hand

$$\nabla_i \partial_j u = \bar{\nabla}_i \partial_j u - \omega^0_{ij} \partial_0 u = \bar{\nabla}_i \partial_j u + N^{-1} K_{ij}$$

The wave map equation reads therefore

$$D_t (N^{-1} \partial_0 u^A) = \bar{\nabla}^i (N \partial_i u^A) + \tau \partial_0 u^A.$$

The commutation relation written for $\partial_0 u$ applies to $u' \equiv N^{-1}\partial_0 u$, we have

$$(D_t \bar{\nabla}_i - \bar{\nabla}_i D_t)(N^{-1} \partial_0 u^A) = N^{-1} R^A_{CD\ E}(u) \partial_0 u^C \partial_i u^B \partial_0 u^E,$$

We have therefore if u is a wave map

$$D_t \bar{\nabla}_i (N^{-1} \partial_0 u^A) = \bar{\nabla}_i \{\bar{\nabla}^j (N \partial_j u^A) + \tau \partial_0 u^A\} + N^{-1} R^A_{CD\ E}(u) \partial_0 u^C \partial_i u^B \partial_0 u^E.$$

Inserting this expression in $D_t I_0$, adding $D_t I_1$ and integrating we find:

$$\int_{S_t} D_t (I_0 + I_1) \mu_t =$$
$$\int_{S_t} \{\bar{\nabla}_i (\bar{\nabla}^j (N \partial_j u) + \tau \partial_0 u) + N^{-1} R^A_{CD\ E}(u) \partial_0 u^C \partial_i u^B \partial_0 u^E$$
$$+ NK_{ij} \bar{\nabla}^j u'\}.\bar{\nabla}^i u' \mu_t + \int_{S_t} \{-D_t \partial_i u.\bar{\nabla}^i \bar{\nabla}^j \partial_j u + F.\bar{\nabla}^j \partial_j u\} \mu_t.$$

The derivatives of third order in u cancel if u is a wave map. Indeed a straightforward computation (recall that $u' \equiv N^{-1}\partial_0 u$ and $D_t \partial_i u = \bar{\nabla}_i \partial_0 u$) gives

$$\bar{\nabla}^i (\bar{\nabla}^j (N \partial_j u).\bar{\nabla}_i u' \text{ - } \bar{\nabla}^i \bar{\nabla}^j \partial_j u.\bar{\nabla}_i \partial_0 u \equiv C$$

with, by elementary computation,

$$C \equiv \text{ - } \bar{\nabla}^i \bar{\Delta} u \partial_i N.u' + (\partial_i N \bar{\Delta} u + \partial^j N \bar{\nabla}_i \partial_j u).\bar{\nabla}^i u'$$

Under integration on S_t this term is equivalent to the following one, denoted B:

$$B \equiv \bar{\Delta} N \bar{\Delta} u.u' + (2\partial_i N \bar{\Delta} u + \partial^j N \bar{\nabla}_i \partial_j u).\bar{\nabla}^i u'$$

We have found:

$$\frac{de^{(1)}}{dt} = \int_{S_t} \{\partial_0(I_0 + I_1) - N\tau(I_0 + I_1)\}\mu_t$$
$$= \int_{S_t}\{ A_1 + A_0 + B - N\tau \ (I_0 + I_1)\mu_t$$

with

$$A_0 \equiv \{\bar\nabla_i(N\tau u') + N^{-1}R^A_{CD\ E}(u)\partial_0 u^C\partial_i u^B\partial_0 u^E \ + \ NK_{ij}\bar\nabla^j u'\}.\bar\nabla^i u'$$

Setting as in the previous section

$$K_{ij} \equiv P_{ij} + \tfrac{1}{n}g_{ij}\tau, \quad \text{with} \quad g^{ij}P_{ij} = 0$$

gives

$$A_0 \equiv 2N(1 + \tfrac{1}{n})\tau \ I_0 + NP_{ij}\bar\nabla^i u'.\bar\nabla^j u' +$$
$$\{\partial_i(N\tau)u' + N^{-1}R^A_{CD\ E}(u)\partial_0 u^C\partial_i u^B\partial_0 u^E \ \}.\bar\nabla^i u'$$

while $A_1 \equiv F.\bar\nabla^j\partial_j u$, given by

$$A_1 \equiv \{R_{CD\ E}\partial_0 u^C\partial^i u^D\partial_i u^E + 2NK^{ij}\bar\nabla_j\partial_i u$$
$$+(-\bar\nabla^h(N\tau) + \mathbf{2}\bar\nabla_i(NK^{ih}))\partial_h u\}.\bar\Delta u$$

can be written:

$$A_1 \equiv N\tau\tfrac{4}{n}I_1 + 2NP^{ij}\bar\nabla_i\partial_j u.\bar\Delta u + \{R_{CD\ E}\partial_0 u^C\partial^i u^D\partial_i u^E$$
$$+[-\bar\nabla^h(N\tau) + \mathbf{2}\bar\nabla_i(NK^{ih})]\partial_h u\}.\bar\Delta u$$

We obtain the following theorem by summing and rearranging the various terms that we have found.

Theorem. (second energy equality). If u is a wave map, its second energy satisfies the following equality.

$$\frac{de^{(1)}(t,u)}{dt} = \int_{S_t}\{ \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}\} \ \mu_t$$

with

$$\mathrm{I} \equiv N\tau \ [(1 + \tfrac{\mathbf{2}}{n}) \ I_0 + (\tfrac{4}{n} - 1)I_1]$$
$$\mathrm{II} \equiv (2\partial_i N\bar\Delta u + \partial^j N\bar\nabla_i\partial_j u).\bar\nabla^i u' + NP_{ij}[\bar\nabla^i u'.\bar\nabla^j u' \ + \ 2\bar\nabla^i\partial^j u.\bar\Delta u]$$
$$\mathrm{III} \equiv \partial^h(N\tau)u'.\bar\nabla_h u' + [(\bar\nabla(N\tau) + \mathbf{2}\bar\nabla_i(NK^{ih}))\partial_h u + \ \bar\Delta N u'].\bar\Delta u$$
$$\mathrm{IV} \equiv N^{-1}R^A_{CD\ E}(u)\partial_0 u^C\partial_i u^B\partial_0 u^E \ .\bar\nabla^i u' + R_{CD\ E}\partial_0 u^C\partial^i u^D\partial_i u^E.\bar\Delta u.$$

We note that the terms I and II are quadratic in the second derivatives of u, with coefficient $N\tau$ for I, up to numbers depending only on n. In the case of II the coefficients belong to DN or NP. The term III is bilinear in the first and the second derivatives of u with coefficients $\bar\nabla(NK)$ and $\bar\Delta N$, while IV is linear in second derivatives of u with coefficients cubic in the first derivatives of u and linear in the Riemann tensor of the target metric h.

We note also the following lemma.

Lemma. For an arbitrary map for which the following integrals make sense the following equality holds:

$$\int_{S_t}\bar\Delta u.\bar\Delta u\mu_t = \int_{S_t}| \ \bar\nabla\partial u \ |^2_{g,h} \ \mu_t \ +$$
$$\int_{S_t}\{\bar R^{ij}\partial_i u.\partial_j u \text{ - } R_{AB\ C}\partial^i u^A\partial^j u^B\partial_i u^C.\partial_j u\}\mu_t$$

where $\bar{R}_{ij}$ is the Ricci tensor of the space metric g.

Proof. Stokes' formula gives

$$\int_{S_t} \bar{\nabla}^i \partial_i u . \bar{\nabla}^j \partial_j u \mu_t = -\int_{S_t} \bar{\nabla}^j \bar{\nabla}^i \partial_i u . \partial_j u \mu_t$$

The Ricci formula gives

$$\bar{\nabla}^j \bar{\nabla}^i \partial_i u \equiv \bar{\nabla}^i \bar{\nabla}^j \partial_i u - \bar{R}^{ih} \partial_h u + R_{AB}{}^{\cdot}{}_C \partial^j u^A \partial^i u^B \partial_i u^C .$$

Another application of Stokes' formula achieves the proof of the lemma.

6 Case of $n \leq 3$

In the case where $n \leq 3$ the Sobolev embedding theorem can be used to estimate the second energy $e^{(1)}(t,u)$ in terms of the H_1 norms of Du and u.

We enunciate and prove a general theorem.

Theorem. (second energy estimate). There exists a number $T > 0$ and a function $C(t)$ continuous in [0,T) such that if **g** satisfies the hypothesis and Riemann(h) is uniformly bounded on the target M then the second energy $y(t)$ satisfies the inequality:

$$y(t) \leq C(t) \quad \text{for} \quad 0 \leq t < T$$

Proof. We first bound the absolute values of the various terms appearing in the right hand side of the energy equality proved in the previous section. We denote generically by C numbers depending only on the dimension n. We denote by $|.|_{\ g,h}$ pointwise norms in the metrics g and h.

We have:

$$|\mathrm{I}| \leq CN \mid \tau \mid (I_0 + I_1).$$

Rather than bounding the absolute value of II we will bound at once its integral. We use the lemma of the previous section which implies

$$\int_{S_t} \mid \bar{\nabla} Du \mid^2_{g,h} \mu_t \leq \int_{S_t} \{ \mid \bar{\Delta} u \mid^{\ 2}_{\ g,h} + \mid \mathrm{Ricci(g)} |_g | \ Du \mid^2_{g,h}$$
$$+ \ |\mathrm{Riemann(h)}|_{\ h} | \ DU \mid^{\ 4}_{\ g,h} \} \mu_t .$$

We use the general property of scalar products that $\mid a.b \mid \leq \mid a \mid \mid b \mid \leq \frac{1}{2} (\mid a \mid^{\ 2} + \mid b \mid^2)$ to obtain

$$\int_{S_t} |\mathrm{II}| \ \mu_t \leq \int_{S_t} | \ II_a \mid \mu_t + \int_{S_t} |\mathrm{II}_b \mid \mu_t$$

with

$$\int_{S_t} |\mathrm{II}_a| \ \mu_t \leq C \int_{S_t} \{ \ [| \ DN \mid_g + N \mid P \mid_g][I_0 + I_1]$$
$$+ \mid DN \mid_{\ g} \mid Ricci(g) \mid_g | \ Du \mid_{g,h} I_0^{1/2} + \mid NP \mid_g | \ Ricci(g) \mid_g | \ Du \mid_{g,h} I_1^{1/2} \ \} \ \mu_t .$$

while

$$\int_{S_t} \mid II_b \mid \mu_t \leq \int_{S_t} \ \{[| \ DN \mid_{\ g} + \mid NP \mid_{\ g}] \ |\mathrm{Riemann(h)}|_{\ h} | \ Du \mid^{\ 4}_{\ g,h} \} \mu_t$$

The absolute value of III is bounded as follows:

$$|\mathrm{III}| \leq C \mid D(N\tau) \mid_g | \ u' \mid_h I_0^{1/2}] + [(| \ D(N\tau) \mid_g + \mid \bar{\nabla}(NK) \mid_g) \mid Du \mid_{g,h} + \mid \bar{\Delta} N \mid | \ u' \mid_h] I_1^{1/2} .$$

Finally

$$|\text{IV}| \leq CN \mid Riemann(h(u)) \mid_h [| Du |^2_{g,h} | u' |_h I_1^{1/2} + | Du |_{g,h} | u' |^2_h I_0^{1/2}]$$

The integrals of I, II_a, III can immediately be bounded, for any n, in terms of the first and second energies of u through the use of the Cauchy-Schwarz inequality, if we suppose that DN, NP, their gradients and the Ricci tensor of g are uniformly bounded in g norm on S_t.

We denote by y the second energy, namely we set:

$$y \equiv e^{(1)}(t,u) \equiv y_0 + y_1$$

with

$$y_0(t) \equiv \int_{S_t} I_0 \mu_t, \quad y_1(t) \equiv \int_{S_t} I_1 \mu_t$$

Recall that the first energy was $e(t,u) \equiv e_0 + e_1$ with

$$e_0(t) \equiv \frac{1}{2} \int_{S_t} |u'|^{\,2}_{h} \mu_t, \quad e_1(t) \equiv \frac{1}{2} \int_{S_t} | Du |^2_{g,h} \mu_t,$$

We then obtain, omitting to write the explicit dependence on t to abbreviate notations and denoting by C constants depending only on n,

$$\int_{S_t} |\text{I}| \mu_t \leq C \, [Sup_{S_t} | N\tau |] [y_0 + y_1]$$

$$\int_{S_t} |\text{II}_a| \mu_t \leq C\{[\text{Sup}_{S_t} \mid DN \mid_g] \, y_0^{1/2} y_1^{1/2} + [\, \text{Sup}_{S_t} | NP \mid_g][y_0 + 2y_1]$$

$$+ [\text{Sup}_{S_t} | DN \mid_g \mid Ricci(g) \mid_g] e_1^{1/2} y_0^{1/2} + [Sup_{S_t} \mid NP \mid_g | Ricci(g) \mid_g] e_1^{1/2} y_1^{1/2} \; .$$

<u>Remark.</u> One can use an L^p norm of Ricci(g) intead of the Sup norm, and estimates of an L^q norms of Du and u'. These norms themselves being estimated in terms of the first and second energies, as we will do later in bounding the integrals of IV and II_b.

We now estimate the integral of III. We find:

$$\int_{S_t} |\text{III}| \mu_t \leq C\{[\text{Sup}_{S_t} \mid D(N\tau) \mid] e_0^{1/2} y_0^{1/2}$$

$$+ \text{Sup}_{S_t} [| D(N\tau \mid_g + \mid \bar{\nabla}(NP) \mid_g] e_1^{1/2} y_1^{1/2} + [\text{Sup}_{S_t} \mid \bar{\Delta} N \mid] e_0^{1/2} y_1^{1/2}.$$

Since IV is cubic in ∂u some further estimates are needed to obtain its bound in terms of e and y. We proceed as follows.

The Cauchy-Schwarz inequality implies:

$$\int_{S_t} |\text{IV}| \mu_t \leq \text{C}[\text{Sup}_{S_t} N \mid Riemann(h(u)) \mid_h][\| | Du |^2_{g,h} | u' |_h \|_{L^2(g)} \, y_1^{1/2}$$

$$+ \| | Du |_{g,h} | u' |^2_h \|_{L^2(g)} \, y_0^{1/2}]$$

By Hölder's inequality we have for arbitrary functions F and G on S:

$$\| F^2 G \|_{L^2(g)} \leq \| F^2 \|_{L^3(g)} \| G \|_{L^6(g)}, \quad \text{because} \quad \frac{1}{2} = \frac{1}{3} + \frac{1}{6} \; .$$

This inequality together with $\| F^2 \|_{L^3} \equiv \| F \|^2_{L^6}$ gives the estimate

$$\int_{S_t} |\text{IV}| \mu_t \leq C \, \text{Sup}_{S_t} \mid Riemann(h(u)) \mid_h \{\| | Du |_{g,h} \|^2_{L^6(g)} \| \, |u'|_h \, \|_{L^6(g)} y_1^{1/2}$$

$$+ \| | Du |_{g,h} \|_{L^6(g)} \| | u' |_h \|^2_{L^6(g)} \, y_0^{1/2}\}.$$

Due to the hypothesis made on the metric g the norms in $L^p(g)$ are equivalent to the norms L^p in the Sobolev regular metric s on S. One can use the Sobolev embedding theorem on (S,e) to estimate the L^6 norm of an arbitrary function F on S in terms of its H_1 norms if $n \leq 3$

$$\| F \|_{L^6} \leq C_s \| F \|_{H_1}, \quad \text{with} \quad \| F \|^2_{H_1} \equiv \| F \|^2_{L^2} + \| DF \|^2_{L^2}$$

where DF is the gradient of the scalar function F. Set

$$F_0 \equiv |u'|_h \equiv \{N^{-2} h_{AB} \partial_0 u^A \partial_0 u^B\}^{1/2}.$$

The gradient of F, a scalar function is independent of the metric of the space, that is $DF \equiv \bar{\nabla} F$. Therefore we can use the Leibnitz rule for covariant derivatives of mappings to obtain:

$$DF_0 = \frac{\bar{\nabla} u'.u'}{|u'|_h} \quad \text{which implies} \quad | DF_0|_g \leq | \bar{\nabla} u' |_{g,h},$$

and, with C_g a number depending only on the equivalence bounds between the metrics g and e,

$$| DF_0 | \leq C_g | \bar{\nabla} u' |_{g,h}, \quad \text{hence} \quad \| DF_0 \|_{L^2} \leq C_g y_0^{1/2}.$$

An analogous reasoning applied to

$$F_1 \equiv | Du |_{g,h}$$

gives

$$\| DF_1 \|_{L^2} \leq C_g y_1^{1/2}.$$

Using these inequalities we obtain a bound in terms of the first energy $e \equiv e(t,u)$ and the second energy $y \equiv e^{(1)}(t,u)$, given by the following estimate (we use the fact that if a and b are positive numbers then $(a+b)^3 \leq 4(a^3+b^3)$)

$$\int_{S_t} |\mathrm{IV}| \, \mu_t \leq C C_g C_h \{e^{3/2} y^{1/2} + y^2\}$$

with

$$C_h \equiv \mathrm{Sup}_{S_t} | \mathit{Riemann}(h(u)) |_h$$

The bound of the integral of $|\mathrm{II}_b|$ is obtained similarly because

$$\int_{S_t} |\mathrm{II}_b | \, \mu_t \leq \int_{S_t} | DN |_g \, C_h \, | Du |^4_{g,h} \, \mu_t$$

$$\leq \mathrm{Sup}_{S_t} [| DN |_g + | NP |_g] C_h \| Du \|_{L^2(g)} \| Du \|^3_{L^6(g)}.$$

Therefore

$$\int_{S_t} |\mathrm{II}_b| \, \mu_t \leq C'_g C_h (e^{1/2} y^{3/2} + e^{3/2} y^{1/2})$$

By the first energy estimate we know that $e \equiv e(t,u)$ is a continuous function of $t \in [0,\infty)$. The obtained inequality give therefore for $y(t)$ a differential inequality of the following type:

$$\frac{dy}{dt} \leq C\{ \alpha y + \beta y^{1/2} + \gamma y^{3/2} + \delta y^2\}$$

The theorem follows from the application of Gromwall's lemma and the fact that the differential equation satisfied by y corresponding to this differential inequality has a continuous solution z on the interval $[0,T)$, for some small enough $T > 0$ which takes the value $z(0) = y(0)$ for $t = 0$.

The expressions for the functions α, β ,γ, δ can be read from the inequalities written above.

The coefficients γ and δ of the nonlinear terms are zero if $C_h = 0$, i.e. if the target is flat. The nonflatness of the target is an obstruction to a global in time estimate.

Remark. The term in $C'_{\mathbf{g}}$ can be expressed differently, using an L^p norm of Ricci(g) intead of the Sup norm, and estimates of an L^q norms of Du and u', estimated again in terms of the first and second energies.

7 Estimate of H_1 Norms

We have seen that the L^2 norms on S_t of DU and $N^{-1}\partial_0 U$ are equal to the energy elements $e_1(t,u)$ and $e_0(t,u)$ respectively. It is not true for the H_1 norms of these quantities compared with the second energy which are defined through covariant mapping derivatives.

For instance we have (cf. Sect. 2)

$$D_i \partial_j U^a \equiv \partial_A i^a D_i \partial_j u^A - K^a_{bc} \partial_i U^b \partial_j U^c.$$

We deduce from this identity and the multiplication properties of Sobolev spaces again an estimate of the H_1 norm of DU on S_t in terms of the first and second energies of u, hence the following lemma.

Lemma. The Cauchy problem for the wave map equation on $S\times\ [\ t_0, \infty)$ has a global solution if its second energy does not blow up in a finite time.

8 Case $n = 1$

In this case the Gagliardo-Nirenberg interpolation inequalities as extended by Aubin 1982 [11] to riemannian manifolds can be used to reduce the degree of the terms in second derivatives appearing in the final estimate. This method was used by Ginibre and Velo (1981) [10] to prove global existence of wave maps on two-dimensional Minkowski space time. However, the interpolation theorem on a compact manifold involves the mean value of the function one wants to estimate, and this poses difficulties. Instead of this interpolation we will use simply the Sobolev embedding theoren of L^3 into W^1_1 when $n = 1$: in this case the Sobolev embedding theorem that there exists a constant C_s, depending only on S and the given metric s, such that

$$\| F \|_{L^3} \leq C_s \| F \|_{W^1_1}, \quad \text{with} \quad \| F \|_{W^1_1} \equiv \| F \|_{L^1} + \| DF \|_{L^1}$$

, where DF is the gradient of the scalar function F. Note that, if a function is in L^6, its square is in L^3. Set

$$F_0 \equiv |u'|^2_h \equiv N^{-2} h_{AB} \partial_0 u^A \partial_0 u^B.$$

The gradient of F, a scalar function is independent of the metric of the space, that is $DF \equiv \bar{\nabla} F$. Therefore we can use the Leibnitz rule for covariant derivatives of mappings to obtain:

$$DF_0 = 2\bar{\nabla} u'.u' \quad \text{which implies} \quad |\ DF_0|_g \leq |\ \bar{\nabla} u' \ |_{g,h}.\ |u'|_h,$$