Asymptotic Behavior of Least Energy Solutions of a Biharmonic Equation in Dimension Four

Mohamed Ben Ayed[a], Khalil El Mehdi[b] and Massimo Grossi[c*†]

a: Département de Mathématiques, Faculté des Sciences de Sfax, Route Soukra, Sfax, Tunisia.

b: Faculté des Sciences et Techniques, Université de Nouakchott, BP 5026, Nouakchott, Mauritania.

c: Dipartimento di Matematica, Universitá di Roma "La Sapienza", P. le A. Moro 2, 00185, Roma, Italy.

Abstract. In this paper we consider a biharmonic equation on a bounded domain in $\mathbb{R}^4$ with large exponent in the nonlinear term. We study asymptotic behavior of positive solutions obtained by minimizing suitable functionals. Among other results, we prove that c_p, the minimum of energy functional with the nonlinear exponent equal to p, is like $\rho_4 e/p$ as $p \to +\infty$, where $\rho_4 = 32\omega_4$ and ω_4 is the area of the unit sphere S^3 in $\mathbb{R}^4$. Using this result, we compute the limit of the L^∞-norm of least energy solutions as $p \to +\infty$. We also show that such solutions blow up at exactly one point which is a critical point of the Robin function. **2000 Mathematics Subject Classification:** 35J60, 35J65.

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1 Introduction and Main Results

Recently, there has been considerable interest in equation involving biharmonic operator Δ^2. A particular feature of biharmonic operator is that it is conformally invariant. More precisely, let (M^4, g) be a smooth 4-dimensional Riemannian manifold, S_g be the scalar curvature of g and Ric_g be the Ricci curvature of g. Then the Paneitz operator defined by

$$P_g^4\varphi = \Delta_g^2\varphi - \operatorname{div}_g(\frac{2}{3}S_g - 2\mathrm{Ric}_g)d\varphi$$

is conformally invariant in the sense that if $\tilde{g} = e^{2u}g$ is a conformal metric to g, then

$$P_{\tilde{g}}^4\varphi = e^{-4u}P_g^4(\varphi) \qquad \text{for all } \varphi \in C^\infty(M),$$

and it can be seen as a natural extension of the conformal Laplacian on 2-manifolds. We refer to [5] for the related topics and their recent developments. Our purpose in this paper is to study the following nonlinear elliptic problem under the Navier boundary condition

$$(P_p) \quad \begin{cases} \Delta^2 u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

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†E-mail addresses: `Mohamed.Benayed@fss.rnu.tn` (M. Ben Ayed), `khalil@univ-nkc.mr` (K. El Mehdi) and `grossi@mat.uniroma1.it` (M. Grossi).

where Ω is a bounded and smooth domain in $\mathbb{R}^4$ and p is a large positive parameter. Since a complete classification of the solutions to (P_p) is still open, we will focus on the solutions obtained by the following variational method.
We define on $E := H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}$ the functional

$$J_p(u) = \frac{\int_\Omega |\Delta u|^2}{(\int_\Omega |u|^{p+1})^{2/(p+1)}}. \tag{1.1}$$

The space E is equipped with the norm $||.||$ and the corresponding inner product $(.,.)$ defined by

$$||u|| = \left(\int_\Omega |\Delta u|^2 \right)^{1/2}, \qquad (u,v) = \int_\Omega \Delta u \Delta v \qquad \text{with} \ \ u,v \in E. \tag{1.2}$$

We consider the following minimizing problem

$$c_p := \inf_{u \in E \setminus \{0\}} J_p(u). \tag{1.3}$$

A standard variational argument shows that c_p is achieved by a positive function. Then, up to a multiplicative constant we find a positive function u_p which solves (P_p) and satisfies

$$c_p = \frac{\int_\Omega |\Delta u_p|^2}{(\int_\Omega |u_p|^{p+1})^{2/(p+1)}}. \tag{1.4}$$

Throughout the rest of this paper we consider solutions of (P_p) obtained in this way. Our goal here is to study the asymptotic behavior of u_p as p goes to infinity. Notice that, such a study for second order elliptic equations in $\mathbb{R}^2$ has attracted considerable attention in last decades, see for example [2], [8], [11], [14], [15], [16], and the references therein. However, as far as the authors know, the case of fourth order elliptic equations in $\mathbb{R}^4$ has not been considered before and this is precisely the first aim of the present paper. One of our results is to obtain the following asymptotic estimate for u_p.

Theorem 1.1 *Let u_p be a solution to (P_p). Then*

$$\lim_{p \to \infty} ||u_p||_\infty = \sqrt{e}. \tag{1.5}$$

where $||.||_\infty$ denotes the L^∞-norm.

To prove Theorem 1.1, we use a blow up technique used by Adimurthi-Grossi [2] and Adimurthi-Struwe [3]. More precisely let us define the function

$$Z_p : \Omega_p := \frac{\Omega - x_p}{\varepsilon_p} \to \mathbb{R}, \ x \mapsto Z_p(x) = \frac{p}{u_p(x_p)} \left(u_p(\varepsilon_p x + x_p) - u_p(x_p) \right), \tag{1.6}$$

where x_p is such that $u_p(x_p) = ||u_p||_\infty$ and where ε_p is such that

$$\varepsilon_p^4 p ||u_p||_\infty^{p-1} = 1. \tag{1.7}$$

Then we have the following result which is the main step in the proof of Theorem 1.1.

Theorem 1.2 *For any sequence Z_{p_n} of Z_p with $p_n \to +\infty$, there exists a subsequence of Z_{p_n}, still denoted by Z_{p_n}, such that $Z_{p_n} \to Z$ in $C^4_{loc}(\mathbb{R}^n)$, where*

$$Z(x) = -4Log\left(1 + \frac{|x|^2}{8\sqrt{6}}\right).$$

To prove Theorem 1.2, we begin by showing the existence of Z satisfying $Z_{p_n} \to Z$ in $C^4_{loc}(\mathbb{R}^n)$ and such that

$$\begin{cases} \Delta^2 Z = e^Z \text{ in } \mathbb{R}^4 \\ e^Z \in L^1(\mathbb{R}^4). \end{cases} \tag{1.8}$$

Let us recall that the corresponding second order equation is

$$\begin{cases} -\Delta Z = e^Z \text{ in } \mathbb{R}^2 \\ e^Z \in L^1(\mathbb{R}^2). \end{cases} \tag{1.9}$$

By employing the method of moving planes, Chen and Li [6] were able to classify all the solutions of (1.9). However equation (1.8) is very different from (1.9). Indeed, in contrast with equation (1.9), a study of radial solutions of (1.8) shows that there are solutions of (1.8) which do not come from the smooth function on S^4 through the stereographic projection. To overcome this difficulty, Lin [12] added the following constraint in the behavior of Z at ∞,

$$|Z(x)| = o(|x|^2) \quad \text{at} \quad \infty. \tag{1.10}$$

Under the constraint (1.10), Lin [12] was able to classify all the solutions of (1.8), (see Theorem 1.1 of [12]). So in order to use Lin's result, we have to prove the estimate (1.10). Generally, the proof of this estimate is quite non trivial, see p 804-807 in [13]. Here, we will give a direct and easy proof of (1.10) in Section 3.

Regarding the proof of Theorem 1.1, it is inspired by some arguments developed in [2] for the corresponding second order equation. However, in our case we have to face the difficulty coming from the fact that the extension by zero outside Ω is not a continuous map from $H^2(\Omega) \cap H^1_0(\Omega)$ to $H^2(\mathbb{R}^n)$. To state our next result, let us mention that the corresponding problem to (P_p) in higher dimensions was studied by Ben Ayed-El Mehdi [4] and Chou-Geng [7]. They considered the following nonlinear problem on a smooth bounded domain Ω in $\mathbb{R}^n$ with $n \geq 5$

$$\begin{cases} \Delta^2 u = u^p,\ u > 0 & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega. \end{cases}$$

They showed that when $p \to (n+4)/(n-4)$, the critical Sobolev exponent for the embedding $H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow L^q(\Omega)$, the solution obtained by the variational method will blow up at some point x_0 which is a critical point of the Robin function R, where $R(x) = H(x,x)$, and H is the regular part of the Green's function of the biharmonic operator Δ^2 with the Navier boundary condition. Our goal in the next result is to prove a similar result for (P_p) when $p \to +\infty$. Namely, we will prove the following theorem.

Theorem 1.3 *For any sequence u_{p_n} of u_p with $p_n \to +\infty$, there exists a subsequence of u_{p_n}, still denoted by u_{p_n}, such that, for any compact set $K \subset \bar{\Omega}\diagdown\{x_0\}$, we have*
i. $p_n u_{p_n} \to \rho_4\sqrt{e}G(.,x_0)$ *in* $C^4(K)$, $p_n\Delta u_{p_n} \to \rho_4\sqrt{e}\Delta G(.,x_0)$ *in* $C^2(K)$, *where* $\rho_4 = 32|S^3|$, $x_0 = \lim_{n\to+\infty} x_{p_n}$ *and* $G(x,y)$ *is the Green's function of* Δ^2 *with the Navier boundary condition :* $\Delta G(x,y) = G(x,y) = 0$ *on* $\partial\Omega$. *We recall that* x_{p_n} *is a point of* Ω *such that* $u_{p_n}(x_{p_n}) = ||u_{p_n}||_\infty$.
ii. *In addition,* x_0 *is a critical point of the Robin function,* R *defined by* $R(x) = H(x,x)$, *where*

$$H(x,y) = G(x,y) + \frac{8}{\rho_4} Log|x-y|$$

is the regular part of the Green's function.

Our paper is organized as follows. In Section 2 we prove some crucial lemmas needed in the proof of our results. Then we prove Theorem 1.2 in Section 3, Theorem 1.1 in Section 4, and Theorem 1.3 in Section 5.

2 Preliminary Results

In this section we prove some auxiliary lemmas needed in the next sections.

Lemma 2.1 *There exists $c > 0$ independent of p such that $||u_p||_\infty \geq c$, where u_p is any solution of (P_p). Furthermore we have*

$$\lim_{p\to+\infty} p||u_p||_\infty^{p-1} = +\infty.$$

Proof. Let λ be the first eigenvalue of Δ^2 under the Navier boundary condition and let φ be a corresponding positive eigenfunction, that is

$$\begin{cases} \Delta^2\varphi = \lambda\varphi \quad \text{in} \quad \Omega \\ \Delta\varphi = \varphi = 0 \,\text{on}\; \partial\Omega. \end{cases}$$

Then for any solution u_p of (P_p), we have

$$0 = \int_\Omega (\Delta^2 u_p \varphi - u_p\Delta^2\varphi) = \int_\Omega u_p\varphi(u_p^{p-1} - \lambda).$$

Since $\varphi > 0$ we have

$$||u_p||_\infty^{p-1} \geq \lambda.$$

Since $\lambda > 0$ our lemma follows. □

Lemma 2.2 *For any $t \geq 2$, there is D_t such that*

$$|u|_t \leq D_t\sqrt{t}||u|| \quad \forall u \in E,$$

where $|.|_t$ denotes the $L^t(\Omega)$-norm.
Furthermore, we have

$$\lim_{t\to+\infty} D_t = (\rho_4 e)^{-1/2}, \quad \textit{with } \rho_4 = 32|S^3|.$$

Proof. We follow the proof of Lemma 2.1 in [14]. On one hand, from Theorem 2 of [1], we have

$$\int_\Omega exp\left(\frac{\rho_4 u^2}{2||u||^2}\right)dx \le c_0.$$

On the other hand, as we have

$$\frac{x^s}{\Gamma(s+1)} \le e^x, \quad \text{for any} \;\; x \ge 0, s \ge 0,$$

we derive that

$$\begin{aligned}\frac{1}{\Gamma(\frac{t}{2}+1)}\int_\Omega u^t dx &= \frac{1}{\Gamma(\frac{t}{2}+1)}\left(\frac{2}{\rho_4}\right)^{\frac{t}{2}}||u||^t\int_\Omega\left(\frac{\rho_4}{2}\frac{u^2}{||u||^2}\right)^{\frac{t}{2}}dx\\ &\le \left(\frac{2}{\rho_4}\right)^{\frac{t}{2}}||u||^t\int_\Omega exp\left(\frac{\rho_4}{2}\frac{u^2}{||u||^2}\right)dx\\ &\le c_0\frac{||u||^t}{(\rho_4/2)^{\frac{t}{2}}}.\end{aligned}$$

Thus

$$\left(\int_\Omega u^t dx\right)^{\frac{1}{t}} \le (c_0)^{\frac{1}{t}}\frac{||u||}{(\rho_4/2)^{\frac{1}{2}}}\left(\Gamma(\frac{t}{2}+1)\right)^{\frac{1}{t}} \le D_t t^{1/2}||u||,$$

where

$$D_t = \frac{c_0^{1/t}}{(\rho_4/2)^{\frac{1}{2}}}\left(\Gamma(\frac{t}{2}+1)\right)^{\frac{1}{t}}t^{-1/2}.$$

Using Stirling's formula, we obtain

$$\left(\Gamma(\frac{t}{2}+1)\right)^{\frac{1}{t}} \sim \left(\left(\frac{\frac{t}{2}}{e}\right)^{t/2}\sqrt{te}e^{\theta_t}\right)^{1/t} \sim \left(\frac{1}{2e}\right)^{1/2}\sqrt{t}, \text{ with } \;\; 0<\theta_t<1/12.$$

Thus we get the desired result. □

Lemma 2.3 *We have that*

$$\lim_{p\to+\infty} pc_p = \rho_4 e, \quad \textit{with} \quad \rho_4 = 32|S^3|.$$

Proof. Without loss of generality we can assume that $0\in\Omega$ and $B(0,2)\subset\Omega$. Let $\varphi\in C_0^\infty(\Omega)$ such that

$$\varphi = 1 \text{ in } \;B(0,1), \quad \varphi = 0 \text{ in } \;\Omega\setminus B(0,2) \text{ and } 0\le\varphi\le 1.$$

We introduce the following function

$$v_p(x) = \varphi(x)\left(1-\frac{4}{p}Log\left(1+\frac{|x|^2}{8\sqrt{6}\eta_p^2}\right)\right) := \varphi(x)w_p(x),$$

where η_p satisfies $p\eta_p^4 e^{(p-1)/2} = 1$.
Of course $v_p \in E$. Let us compute $|v_p|_{p+1}^2$ and $||v_p||^2$.
We have that

$$\int_\Omega |v_p(x)|^{p+1}dx = \int_{B(0,1)} |w_p(x)|^{p+1}dx + \int_{B(0,2)\setminus B(0,1)} \varphi(x)|w_p(x)|^{p+1}dx := I_1 + I_2.$$

Observe that

$$I_1 = \eta_p^4 \int_{B(0,\frac{1}{\eta_p})} \left|1 - \frac{4}{p}\log\left(1 + \frac{|y|^2}{8\sqrt{6}}\right)\right|^{p+1} dy \tag{2.1}$$

For p and R large enough, we have

$$\int_{B(0,R)} \left|1 - \frac{4}{p}\log\left(1 + \frac{|y|^2}{8\sqrt{6}}\right)\right|^{p+1} dy = \int_{B(0,R)} \frac{dy}{\left(1 + \frac{|y|^2}{8\sqrt{6}}\right)^4} + o(1) = \rho_4 + o(1), \tag{2.2}$$

where we have used the fact that $\int_{\mathbb{R}^4} \left(1 + \frac{|y|^2}{8\sqrt{6}}\right)^{-4} dy = \rho_4$ as easy computations show.
On the other hand, it is easy to see that $|1 + x/p|^p \le e^x$ for any $x \in [-\alpha_0 p, 0]$, where $\alpha_0 > 1$ is a fixed constant independent of p. We now notice that, for $y \in B(0, 1/\eta_p)$, we have

$$\begin{aligned}
-4\log\left(1 + |y|^2(8\sqrt{6})^{-1}\right) &\ge -4\log\left(1 + (8\sqrt{6}\eta_p^2)^{-1}\right) \\
&\ge -4\log\left(\eta_p^{-2}\right) - 4\log\left(\eta_p^2 + (8\sqrt{6})^{-1}\right) \\
&\ge -\log\left(p^2 e^{p-1}\right) = -p\left(1 + p^{-1}\log p^2 - p^{-1}\right).
\end{aligned}$$

Thus

$$\int_{B(0,\frac{1}{\eta_p})\setminus B(0,R)} \left|1 - \frac{4}{p}\log\left(1 + \frac{|y|^2}{8\sqrt{6}}\right)\right|^{p+1} dy \le \int_{B(0,\frac{1}{\eta_p})\setminus B(0,R)} \frac{dy}{\left(1 + \frac{|y|^2}{8\sqrt{6}}\right)^4} + o(1) = o(1), \tag{2.3}$$

for R large enough.
Clearly (2.1)-(2.3) imply that

$$I_1 = \rho_4 \eta_p^4 (1 + o(1)).$$

Notice that, for $x \in B(0,2) \setminus B(0,1)$

$$1 - \frac{4}{p}\log\left(1 + \frac{|x|^2}{8\sqrt{6}\eta_p^2}\right) = 1 - \frac{1}{p}\log\left(\frac{1}{\eta_p^8}\right) + O\left(\frac{1}{p}\right) = -\frac{1}{p}\log\left(p^2\right) + O\left(\frac{1}{p}\right).$$

Thus

$$\left|1 - \frac{4}{p}\log\left(1 + \frac{|x|^2}{8\sqrt{6}\eta_p^2}\right)\right| \le e^{-2} \quad \text{for } p \text{ large.}$$

Therefore

$$|I_2| \le ce^{-2(p+1)} = o\left(\eta_p^4\right).$$

Thus

$$\left(\int_\Omega |v_p(x)|^{p+1}dx\right)^{2/(p+1)} = \left(\rho_4\eta_p^4\right)^{2/(p+1)}(1+o(1)) = e^{-1}+o(1).$$

Now, we are going to estimate $||v_p||^2$. To this aim, we write

$$||v_p||^2 = \int_\Omega \left(\varphi^2|\Delta w_p|^2 + |w_p\Delta\varphi|^2 + 4(\nabla\varphi\nabla w_p)^2 + 2w_p\varphi\Delta\varphi\Delta w_p\right) + \int_\Omega \left(4\varphi\Delta w_p\nabla\varphi\nabla w_p + 4w_p\Delta\varphi\nabla\varphi\nabla w_p\right) \tag{2.4}$$

and we have to estimate each term of the right hand-side of (2.4).
First, we observe that

$$\begin{aligned}\int_\Omega \varphi^2|\Delta w_p|^2 &= \int_{B(0,1)}|\Delta w_p|^2 + \int_{B(0,2)\setminus B(0,1)}\varphi^2|\Delta w_p|^2\\ &= \frac{16^2}{p^2}|S^3|\int_0^1 \frac{(r^2+16\sqrt{6}\eta_p^2)^2}{(8\sqrt{6}\eta_p^2+r^2)^4}r^3dr + O\left(\frac{1}{p^2}\right)\\ &= \frac{16^2}{2p^2}|S^3|\left(Log(\eta_p^{-2})+O(1)\right) + O\left(\frac{1}{p^2}\right)\\ &= \frac{\rho_4}{p}(1+o(1)).\end{aligned}$$

For the other terms, one can easy check that

$$|w_p|_{L^\infty(B(0,2)\setminus B(0,1))} = O\left(\frac{1}{p}\right),\ |\nabla w_p|_{L^\infty(B(0,2)\setminus B(0,1))} = O\left(\frac{1}{p}\right),$$

and

$$|\Delta w_p|_{L^\infty(B(0,2)\setminus B(0,1))} = O\left(\frac{1}{p}\right),$$

and thus we derive that all the other terms in the right hand-side of (2.4) are $O\left(\frac{1}{p^2}\right)$. Hence we obtain

$$||v_p||^2 = \frac{\rho_4}{p}(1+o(1)).$$

Thus

$$p\frac{\int|\Delta v_p|^2}{|v_p|^2_{L^{p+1}}} = \rho_4 e(1+o(1))$$

and then

$$pc_p \le \rho_4 e(1+o(1)).$$

Combining this with Lemma 2.2, we obtain

$$\lim_{p\to+\infty} pc_p = \rho_4 e.$$

□ In addition, since u_p satisfies (P_p), we have

$$c_p = \frac{||u_p||^2}{|u_p|_{p+1}^2} = |u_p|_{p+1}^{p-1}$$

and thus we derive from Lemma 2.3 the following result

Corollary 2.4 *We have*

$$\lim_{p\to+\infty} p\int_\Omega u_p^{p+1} = \rho_4 e \ \text{and} \ \lim_{p\to+\infty} p\int_\Omega |\Delta u_p|^2 = \rho_4 e,$$

where $\rho_4 = 32|S^3|$.

Now, we set

$$v_p(x) = pu_p(x), \quad x \in \Omega. \tag{2.5}$$

Observe that v_p satisfies

$$\begin{cases} \Delta^2 v_p = \mu_p v_p^p,\ v_p > 0 & \text{in } \Omega \\ \Delta v_p = v_p = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

where $\mu_p = p^{1-p}$. By Corollary 2.4, we have

$$\mu_p \int_\Omega v_p^p \le c \tag{2.7}$$

for some positive constant c independent of p. By the maximum principle, we have

$$\Delta v_p(x) < 0 \quad \text{for } x \in \Omega.$$

Next, our aim is to prove that the blowup points of v_p are in the interior of Ω. The crucial step is to estimate v_p in the region away from the blowup points. Set

$$\overline{\Omega}\setminus\Gamma = \{x \in \overline{\Omega} \mid \exists r_0 > 0 \text{ such that } \mu_p \int_{B(x,r_0)\cap\Omega} v_p^p(y)dy < \varepsilon_0,\ \forall p\}, \tag{2.8}$$

where ε_0 is a small fixed positive number. Thus we have:

Lemma 2.5 *Let K be a compact set, $K \subset \overline{\Omega}\setminus\Gamma$. Then we have*

$$p||u_p||_{L^\infty(K)}^{p-1} \le C, \quad \text{and} \quad ||\Delta v_p||_{L^\infty(K)} \le C, \quad ||v_p||_{L^\infty(K)} \le C,$$

where C is a positive constant independent of p.

Proof. Since K is a compact set in $\overline{\Omega}\setminus\Gamma$, we have

$$\exists\, \alpha > 0 \text{ such that } d(\Gamma, K) \ge 2\alpha > 0. \tag{2.9}$$

Using equation (P_p), we derive that

$$u_p(y) = \int_\Omega G(x,y)u_p^p(x)dx, \quad \forall\, y \in K,$$

where G is the Green's function of Δ^2 defined in Theorem 1.3.
Thus

$$u_p(y) = \int_\Omega H(x,y)u_p^p(x)dx - \frac{8}{\rho_4}\int_\Omega Log|x-y|u_p^p(x)dx = I_1(y) + \frac{8}{\rho_4}I_2(y), \quad \forall y \in K, \quad (2.10)$$

where H is the regular part of G.
We need to estimate each integral in (2.10). To this aim, suppose that the following claims are true.
Claim 1. We have that

$$I_1(y) \leq \frac{c}{p}, \quad \forall\ y \in K,$$

where c is a positive constant.
Claim 2. We have that

$$I_2(y) \leq \rho_4/32 \quad \forall\ y \in K.$$

Using Claims 1 and 2 and (2.10), we derive that

$$u_p(y) \leq 1/2, \quad \forall\ y \in K.$$

Then we obtain

$$p||u_p||^{p-1}_{L^\infty(K)} \leq pe^{(p-1)Log(1/2)} \to 0 \text{ when } p \to \infty. \quad (2.11)$$

This completes the proof of the first part of Lemma 2.5 under Claims 1 and 2.
Now we will prove the second part of Lemma 2.5. Let G_Δ be the Green's function of Δ with Dirichlet condition. We have, for any $y \in K$

$$-p\Delta u_p(y) = p\int_\Omega G_\Delta(x,y)u_p^p(x)dx. \quad (2.12)$$

Observe that for $y \in K$, $\overline{\Omega} \cap \overline{B(y,\alpha)}$ is a compact set of $\overline{\Omega} \setminus \Gamma$, then (2.11) implies that

$$p\int_{\Omega\cap B(y,\alpha)} G_\Delta(x,y)u_p^p(x)dx \leq c\int_{\Omega\cap B(y,\alpha)} G_\Delta(x,y)dx \leq c\int_{B(y,\alpha)} \frac{dx}{|x-y|^2} \leq c. \quad (2.13)$$

Now, in $\Omega \setminus B(y,\alpha)$ we have $G_\Delta(x,y) \leq c$. Thus

$$p\int_{\Omega\setminus B(y,\alpha)} G_\Delta(x,y)u_p^p(x)dx \leq cp\int u_p^p(x)dx \leq c, \quad (2.14)$$

where we have used Corollary 2.4 in the last inequality.
Then, using (2.12), (2.13) and (2.14), we derive that

$$p|\Delta u_p(y)| \leq c \quad \forall\ y \in K.$$

Lastly, we write

$$v_p(y) = \int_\Omega G(x,y)pu_p^p(x)dx$$

and as above, we deduce that $\| v_p \|_{L^\infty(K)} \leq C$. This completes the proof of Lemma 2.5 under Claims 1 and 2. It now suffices to prove Claims 1 and 2.

Proof of Claim 1. For any $y \in K$, we have

$$\begin{aligned} I_1(y) &= \int_\Omega H(x,y) u_p^p(x) dx = \int_\Omega H(x,y) \Delta^2 u_p(x) dx \\ &= \int_\Omega \Delta H(x,y) \Delta u_p(x) dx + \int_{\partial\Omega} H(x,y) \frac{\partial \Delta u_p}{\partial \nu}(x) dx. \end{aligned} \tag{2.15}$$

Observe that H satisfies

$$\Delta^2 H = 0 \text{ in } \Omega, \quad -\Delta H < 0 \text{ on } \partial\Omega.$$

Thus $-\Delta H < 0$ in Ω. Furthermore, $-\Delta u_p > 0$ in Ω. Then

$$\int_\Omega \Delta H \Delta u_p < 0. \tag{2.16}$$

For the second integral, we have $\partial(\Delta u_p)/\partial\nu > 0$ (since $-\Delta u_p > 0$ in Ω and $\Delta u_p = 0$ on $\partial\Omega$). Furthermore, since $H(x,y) = (8/\rho_4) Log|x-y|$ on $\partial\Omega$, we have

$$\int_{\partial\Omega \cap B(y,1)} H(x,y) \frac{\partial \Delta u_p}{\partial \nu}(x) dx < 0. \tag{2.17}$$

Using the Green's formula and Corollary 2.4, we obtain

$$\int_{\partial\Omega \setminus B(y,1)} H(x,y) \frac{\partial \Delta u_p}{\partial \nu}(x) dx \leq c \int_{\partial\Omega} \frac{\partial \Delta u_p}{\partial \nu}(x) dx = c \int_\Omega \Delta^2 u_p = c \int_\Omega u_p^p \leq \frac{c}{p}. \tag{2.18}$$

Thus

$$I_1(y) \leq c/p, \quad \forall y \in K.$$

The proof of Claim 1 is thereby completed.

Proof of Claim 2. For any $y \in K$, we have

$$I_2(y) = \int_\Omega -Log|x-y| u_p^p(x) dx,$$

Since K is a compact set of $\overline{\Omega} \setminus \Gamma$, we can choose $r_0 = r_0(K)$ such that for every $y \in K$

$$\mu_p \int_{B(y,r_0)} v_p^p(x) dx < \varepsilon_0, \quad \text{for } p \text{ large}.$$

Now, let $r_1 = \min(r_0, \alpha)$ where α is defined by (2.9).

In $\Omega \setminus B(y, r_1)$, we have $|Log|x-y|| \leq c$, where c is a positive constant which depend only on Ω and r_1. Thus

$$\left| \int_{\Omega \setminus B(y,r_1)} Log|x-y| u_p^p(x) dx \right| \leq c \int_\Omega u_p^p \leq c/p. \tag{2.19}$$

For the other integral, using Holder's inequality, we derive that

$$\int_{\Omega\cap B(y,r_1)} -Log|x-y|u_p^p(x)dx \le \left(\int_{\Omega\cap B(y,r_1)} u_p^{p+1}\right)^{\frac{p}{p+1}} \left(\int_{B(y,r_1)} |Log|x-y||^{p+1}\right)^{\frac{1}{p+1}}. \quad (2.20)$$

Since $y \in \overline{\Omega}\setminus\Gamma$, then we have

$$\int_{B(y,r_1)\cap\Omega} u_p^p = \frac{c}{p}\mu_p \int_{B(y,r_1)\cap\Omega} v_p^p \le \frac{c}{p}\varepsilon_0. \quad (2.21)$$

Now, since we can assume that $r_1 < 1$, we have

$$\int_{B(y,r_1)} (-Log|x-y|)^{p+1}dx \le \int_{B(y,1)} (-Log|x-y|)^{p+1}dx = w_4\int_0^1 (-Logr)^{p+1}r^3dr. \quad (2.22)$$

Observe that

$$\int_0^1 (-Logr)^{p+1}r^3dr = \int_0^{+\infty} e^{-4t}t^{p+1}dt = \frac{\Gamma(p+2)}{4^{p+2}}.$$

Using Stirling's formula, we derive that

$$\left(\int_{B(y,r_1)} (-Log|x-y|)^{p+1}\right)^{1/(p+1)} = O(p). \quad (2.23)$$

(2.20), (2.21) and (2.23) imply

$$\int_{\Omega\cap B(y,r_1)} -Log|x-y|u_p^p(x)dx \le C\varepsilon_0. \quad (2.24)$$

(2.19) and (2.24) complete the proof of Claim 2. □

Lemma 2.6 *Let* Γ *be the set defined by* (2.8). *Then we have*

$$\Gamma \subset \Omega.$$

Proof. Using Lemma 2.5, our lemma can be shown by using a local version of the method of moving planes, as done exactly in the same way in [13] (see pages 790-793). □ Clearly,

Lemmas 2.1 and 2.5 imply the following Corollary:

Corollary 2.7 *Let* x_p *be a point of* Ω *such that* $u_p(x_p) = ||u_p||_\infty$. *Thus, for any sequence* x_{p_n} *of* x_p *with* $p_n \to +\infty$, *there exists a subsequence of* x_{p_n}, *still denoted by* x_{p_n}, *such that*

$$\lim_{n\to+\infty} x_{p_n} = x_o \in \Gamma \subset \Omega.$$

3 Proof of Theorem 1.2

We begin by proving the following crucial lemma:

Lemma 3.1 *Let Z_p be the function defined in* (1.6). *Then, for any sequence Z_{p_n} of Z_p with $p_n \to +\infty$, there exists a subsequence of Z_{p_n}, still denoted by Z_{p_n}, such that $Z_{p_n} \to Z$ in $C^4_{loc}(\mathbb{R}^n)$, where Z satisfies*

$$\begin{cases} \Delta^2 Z = e^Z & in \;\; \mathbb{R}^4 \\ Z(0) = 0, \;\; Z \le 0 & in \;\; \mathbb{R}^4 \\ \int_{\mathbb{R}^4} e^Z dx < \infty. \end{cases}$$

Proof. Observe that $\lim_{n\to+\infty} \varepsilon_{p_n} = 0$ (see Lemma 2.1). Now by Corollary 2.7, we know that $\Omega_n := \frac{\Omega - x_{p_n}}{\varepsilon_{p_n}}$ "converges" to $\mathbb{R}^4$ as $n \to +\infty$. Let us write down the equation satisfied by Z_n, where

$$Z_n = \frac{p_n}{u_{p_n}(x_{p_n})}\left(u_{p_n}(\varepsilon_{p_n} x + x_{p_n}) - u_{p_n}(x_{p_n})\right);$$

$$\begin{cases} \Delta^2 Z_n = \left(1 + \frac{Z_n}{p_n}\right)^{p_n} & \text{in } \;\Omega_n \\ 0 < 1 + \frac{Z_n}{p_n} \le 1 & \text{in } \;\Omega_n \\ Z_n = -p_n, \Delta Z_n = 0 & \text{on } \;\partial\Omega_n. \end{cases} \tag{3.1}$$

We want to pass to the limit in (3.1).
Let $B(0,R)$ be the ball of radius R centered at the origin, and let ω_n be the solution of

$$\begin{cases} \Delta^2 \omega_n = \left(1 + \frac{Z_n}{p_n}\right)^{p_n} & \text{in } \;B(0,R) \\ \Delta\omega_n = \omega_n = 0 & \text{on } \;\partial B(0,R). \end{cases} \tag{3.2}$$

By the maximum principle and standard regularity theory, we have that

$$0 \le \omega_n \le c, \quad \text{and} \quad 0 < -\Delta\omega_n \le c,$$

where c is a positive constant independent of n.
In the same way (3.1) implies that

$$0 < -\Delta Z_n \le c,$$

where c is a positive constant independent of n.
Thus $g_n := -\Delta Z_n + \Delta\omega_n$ satisfies

$$0 < g_n \le c, \quad \text{with}\, c \,\text{independent of}\, n.$$

For $x \in B(0,R)$, we set

$$\psi_n(x) = Z_n(x) - \omega_n(x).$$

Note that $\psi_n < 0$ and let us define φ_n by

$$\begin{cases} -\Delta\varphi_n = -\Delta\psi_n = g_n & \text{in } \;B(0,R) \\ \varphi_n = 0 & \text{on } \;\partial B(0,R). \end{cases}$$

Thus

$$0 < \varphi_n \le c, \quad \text{with}\, c\, \text{independent of}\, n.$$

Observe that $(\varphi_n - \psi_n)$ is a positive harmonic function. Hence by Harnack inequality [9], we have the alternative
either
(i) a subsequence of $(\varphi_n - \psi_n)$ is bounded in $L^\infty_{loc}(B(0,R))$
or
(ii) $(\varphi_n - \psi_n)$ converges uniformly to $+\infty$ on compact subsets of $B(0,R)$.
Since $\varphi_n(0) - \psi_n(0) = \varphi_n(0) + \omega_n(0) \le c$, thus case (ii) cannot occur. Hence up to a subsequence, which we denote again by $\varphi_n - \psi_n$, we have $\varphi_n - \psi_n$ is bounded in $L^\infty_{loc}(B(0,R))$ for any $R > 0$ and the same holds for Z_n.
From (3.1), and the standard regularity, we derive that Z_n is bounded in $C^4_{loc}(\mathbb{R}^4)$ and then it converges to a function $z \in C^4(\mathbb{R}^4)$. Passing to the limit in (3.1), we get that Z satisfies

$$\Delta^2 Z = e^Z,\ Z \le 0 \quad \text{in } \mathbb{R}^4 \quad \text{and } Z(0) = 0.$$

Now it remains to prove that

$$\int_{\mathbb{R}^4} e^Z dx < +\infty.$$

Since $Z_n \to Z$ in $C^4_{loc}(\mathbb{R}^4)$, we have

$$\left(1 + \frac{Z_n}{p_n}\right)^{p_n} \to e^Z \quad \text{pointwise in } \mathbb{R}^4.$$

By Fatou's Lemma, we derive that

$$\int_{\mathbb{R}^4} e^Z \le \lim_{n\to+\infty} \int_{\Omega_n} \left(1 + \frac{Z_n}{p_n}\right)^{p_n} dx = \lim_{n\to+\infty} \int_\Omega \frac{u_{p_n}^{p_n}(x)}{|u_{p_n}|_\infty^{p_n}} \frac{dx}{\varepsilon_{p_n}^4}.$$

Using Corollary 2.4 and Lemma 2.1, we derive that $\int_{\mathbb{R}^4} e^Z dx < +\infty$. □ Next we prove Theorem 1.2.

Proof of Theorem 1.2 Using Theorem 1.1 of [12] and Lemma 3.1, we see that it is sufficient to prove the following estimate

$$Z(x) = o(|x|^2) \quad \text{at } \infty.$$

On one hand, by Theorem 1.2 of [12] and Lemma 3.1, we have

$$\Delta Z(0) = -\frac{1}{4\pi^2} \int_{\mathbb{R}^4} \frac{e^{Z(x)}}{|x|^2} dx - 2\sum_{i=1}^4 a_i, \tag{3.3}$$

where $a_i \ge 0$ are constants.
On the other hand, letting $G_{\Delta,n}$ the Green's function of Laplacian operator with Dirichlet boundary condition defined on Ω_n, we have

$$-\Delta Z_n(0) = \frac{1}{4\pi^2} \int_{\Omega_n} G_{\Delta,n}(x,0) \left(1 + \frac{Z_n}{p_n}\right)^{p_n} dx,$$

where $Z_n = Z_{p_n}$, $\Omega_n = \Omega_{p_n}$.
Observe that, for any $R > 0$, we have

$$\begin{aligned}\int_{\Omega_n \setminus B(0,R)} G_{\Delta,n}(0,x)\left(1+\frac{Z_n}{p_n}\right)^{p_n} &\leq \frac{1}{R^2}\int_{\Omega_n \setminus B(0,R)}\left(1+\frac{Z_n}{p_n}\right)^{p_n}\\ &\leq \frac{1}{R^2}\frac{1}{||u_{p_n}||_\infty^{p_n}\varepsilon^4}\int_\Omega u_{p_n}^{p_n}(x)dx\\ &\leq \frac{C}{R^2}p_n\int_\Omega u_{p_n}^{p_n}(x)dx \leq \frac{C_1}{R^2},\end{aligned}$$

where we have used in the last inequality Corollary 2.4 and where C and C_1 are positive constants.
We also have

$$\int_{B(0,R)} G_{\Delta,n}(0,x)\left(1+\frac{Z_n}{p_n}\right)^{p_n} = \int_{B(0,R)} G_{\Delta,n}(0,x)e^Z(1+o(1)).$$

But we have

$$\begin{aligned}\int_{B(0,R)} G_{\Delta,n}(0,x)e^Z &= \int_{B(0,R)}\frac{e^Z}{|x|^2}dx - \int_{B(0,R)} H_{\Delta,n}(0,x)e^Z dx\\ &= \int_{\mathbb{R}^4}\frac{e^Z}{|x|^2}dx + o(1) + O\left(\varepsilon_{p_n}^2\right),\end{aligned}$$

where we have used in the last equality the fact that $0 < H_{\Delta,n}(y,x) \leq \frac{1}{d(y,\partial\Omega_n)^2}$. $H_{\Delta,n}$ denotes here the regular part of $G_{\Delta,n}$.
Thus we obtain

$$-\Delta Z_n(0) = \frac{1}{4\pi^2}\int_{\mathbb{R}^4}\frac{e^Z}{|x|^2}dx + o(1) + O\left(\varepsilon_{p_n}^2\right). \tag{3.4}$$

Using (3.3), (3.4) and the fact that $-\Delta Z_n(0)$ converges to $-\Delta Z(0)$, we derive that

$$a_i = 0, \quad \forall i \in \{1, ..., 4\}.$$

Thus, using the expansion of Z in Theorem 1.2 of [12], we clearly derived the desired behavior of Z at ∞ and therefore our result follows. □

4 Proof of Theorem 1.1

In this section we want to prove Theorem 1.1. To this aim we begin by proving some auxiliary lemmas.

Lemma 4.1 *We have that*

$$\lim_{n\to+\infty} Sup|u_{p_n}|_\infty \leq \sqrt{e}.$$

Proof. We have

$$p_n \int_\Omega u_{p_n}^{p_n+1} = p_n |u_{p_n}|_\infty^{p_n+1} \int_{\Omega_n} \left(1+\frac{Z_n}{p_n}\right)^{p_n+1} \varepsilon_{p_n}^4\, dx$$
$$= |u_{p_n}|_\infty^2 \int_{\Omega_n} \left(1+\frac{Z_n}{p_n}\right)^{p_n+1} dx.$$

Setting $L = \lim_{n\to+\infty} Sup|u_{p_n}|_\infty$ and using Fatou's Lemma and Corollary 2.4, we obtain

$$\rho_4 e \geq L^2 \int_{\mathbb{R}^4} e^Z = L^2\rho_4.$$

Thus our claim follows. □ Now let us consider the linearized operator associated to (P_p), that is $L_p : H^2(\Omega)\cap H_0^1(\Omega) \to H^{-2}(\Omega)$ such that

$$L_p = \Delta^2 - p u_p^{p-1}(x) I, \quad x \in \Omega$$

and let us denote by $\lambda_1(L_p)$, $\lambda_2(L_p)$ the first and the second eigenvalue of L_p. Let us recall a propriety of $\lambda_2(L_p)$.

Lemma 4.2 *We have that*

$$\lambda_2(L_p) \geq 0.$$

Proof. We have

$$c_p := \frac{||u_p||^2}{|u_p|_{p+1}^2} = \inf\{||u||^2 / u \in E, |u|_{p+1} = 1\}.$$

Thus the proof is standard since $\frac{u_p}{|u_p|_{p+1}}$ is a minimizer of F such that

$$F(u) = ||u||^2 \quad \text{with} \quad \int_\Omega |u|^{p+1} = 1.$$

□ Now, for $D \subset \Omega_p$, we consider $L_{p,D} : E \to H^{-2}(D)$, such that

$$L_{p,D} = \Delta^2 - \frac{u_p^{p-1}(\varepsilon_p x + x_p)}{u_p^{p-1}(x_p)} I, \quad x \in D$$

and let us denote by $\lambda_1(L_{p,D})$, $\lambda_2(L_{p,D})$ the first and the second eigenvalue of $L_{p,D}$.

Lemma 4.3 *We have that*

$$\lambda_2(L_{p,\Omega_p}) \geq 0.$$

Proof. Using the scaling $x \mapsto \varepsilon_p x + x_p$, we get $\lambda_2(L_{p,\Omega_p}) = \varepsilon_p^4 \lambda_2(L_p)$ and the result follows by Lemma 4.2. □

Lemma 4.4 *There exists $R_1 > 0$ such that*

$$\lambda_1(L_{p_n, B_{R_1}}) < 0$$

for p_n large enough. B_{R_1} denotes here $B(0, R_1)$.

Proof. For $R > 0$, let us consider the following function $w_p : B_R \to \mathbb{R}$ such that

$$w(x) = 4Log\left(\frac{8\sqrt{6}+R^2}{8\sqrt{6}+|x|^2}\right).$$

Observe that $w = 0$ on ∂B_R and thus $w \in H^2(B_R) \cap H_0^1(B_R)$.
We need to prove the following :

$$\exists R_1 > 0 \quad \text{such that} \qquad I := \int_{B_{R_1}} L_{p,B_{R_1}} w.w < 0, \tag{4.1}$$

for p large enough.
To this aim, we write

$$\begin{aligned} I &= \int_{B_R} \Delta^2 w.w - \int_{B_R} \frac{u_p^{p-1}(\varepsilon_p x + x_p)}{|u_p|_\infty^{p-1}} w^2(x)dx \\ &= \int_{B_R} |\Delta w|^2 - \int_{\partial B_R} \Delta w.\frac{\partial w}{\partial \nu} - \int_{B_R} \frac{u_p^{p-1}(\varepsilon_p x + x_p)}{|u_p|_\infty^{p-1}} w^2(x)dx \\ &:= I_1 - I_2 - I_3. \end{aligned} \tag{4.2}$$

We are going to estimate each integral of the right-hand side of (4.2).
First, we observe that

$$-\Delta w(x) = 16\frac{16\sqrt{6}+|x|^2}{\left(8\sqrt{6}+|x|^2\right)^2} \quad \text{and } \forall x \in \partial B_R,\ \frac{\partial w}{\partial \nu}(x) = \nabla w.\frac{x}{|x|} = \frac{-8R}{8\sqrt{6}+R^2}.$$

Thus

$$I_2 = \int_{\partial B_R} 128\frac{16\sqrt{6}+R^2}{\left(8\sqrt{6}+R^2\right)^3} R dx = 128\omega_4(1 + o_R(1)), \tag{4.3}$$

where ω_4 is the area of the unit sphere S^3 in $\mathbb{R}^4$ and $o_R(1) \to 0$, when $R \to +\infty$.
Secondly, we have

$$I_1 = (16)^2 \int_{B_R} \frac{\left(16\sqrt{6}+|x|^2\right)^2}{\left(8\sqrt{6}+|x|^2\right)^4} dx = (16)^2\omega_4 Log(R)(1 + o_R(1)). \tag{4.4}$$

We also have

$$I_3 = \int_{B_R} \left(1 + \frac{Z_p(x)}{p}\right)^{p-1} w^2(x)dx = \int_{B_R} e^Z w^2(x)dx + o(1),$$

where we have used Theorem 1.2.
Thus, using the expression of Z, we obtain

$$
\begin{aligned}
I_3 &= 16\int_{B_R} \frac{(8\sqrt{6})^4}{\left(8\sqrt{6}+|x|^2\right)^4} Log^2\left(\frac{8\sqrt{6}+|x|^2}{8\sqrt{6}+R^2}\right) dx + o(1)\\
&= 16\omega_4 \int_0^R \frac{(8\sqrt{6})^4 r^3}{\left(8\sqrt{6}+r^2\right)^4}\Big(Log(8\sqrt{6}+r^2) - Log(8\sqrt{6}+R^2)\Big)^2 dr + o(1)\\
&= \frac{4}{3}\omega_4 (8\sqrt{6})^2 \Big(Log^2(8\sqrt{6}+R^2)\Big)(1+o(1)). \qquad (4.5)
\end{aligned}
$$

Combining (4.3), (4.4) and (4.5), we obtain

$$I = -\frac{4\omega_4}{3}(8\sqrt{6})^2 \Big(Log^2(8\sqrt{6}+R^2)\Big)(1+o(1)).$$

This implies (4.1) and therefore our lemma follows. □

Lemma 4.5 *Let p_n large enough such that Lemma 4.4 holds. Then, for p_n large enough, we have*

$$\lambda_1(L_{p_n,\Omega_{p_n}\setminus B(0,R_1)}) > 0.$$

Proof. Arguing by contradiction, we suppose that $\lambda_1(L_{p_n,\Omega_{p_n}\setminus B(0,R_1)}) \le 0$. Then, from Lemma 4.4, $\lambda_1(L_{p_n,B(0,R_1)}) < 0$ for p_n large and hence $\lambda_2(L_{p_n,\Omega_{p_n}}) < 0$. This gives a contradiction with Lemma 4.3. □

Remark 4.6 *Lemma 4.5 implies that the operator $L_{p_n,\Omega_{p_n}\setminus B(0,R_1)}$ satisfies the maximum principle in $\Omega_{p_n}\setminus B(0,R_1)$.*

Now we are in the position to prove Theorem 1.1.
Proof of Theorem 1.1 By Lemma 4.1 we know that (up to subsequence)

$$\lim_{n\to+\infty} |u_{p_n}|_\infty \le \sqrt{e}.$$

Arguing by contradiction let us suppose that there exists a subsequence of u_{p_n}, still denoted by u_{p_n}, such that

$$\lim_{n\to+\infty} |u_{p_n}|_\infty < \sqrt{e}. \qquad (4.6)$$

Now, we will show that for large p_n (4.6) implies the following estimate

$$Z_n(x) \le C + Z(x), \quad \forall x \in \Omega_n, \qquad (4.7)$$

where C is a constant independent of n, $Z_n = Z_{p_n}$ and $\Omega_n = \Omega_{p_n}$.
By Theorem 1.2, $Z_n \to Z$ in $C^4(\overline{B(0,R_1)})$ and hence (4.7) holds for $x \in B(0,R_1)$. Thus it is

enough to prove (4.7) for $x \in \Omega_n \diagdown B(0,R_1)$. To prove this, let us observe that the function Z satisfies

$$\Delta^2 Z = e^Z \geq \left(1+\frac{Z}{p}\right)^p \quad \forall p>1.$$

Now let us consider $\psi_n = Z_n - Z$ in Ω_n.
Observe that :
- if $x \in \partial\Omega_n$, we have

$$\begin{aligned}\psi_n(x) &= -p_n + 4Log\left(1+\frac{|x|^2}{8\sqrt{6}}\right)\\ &= -p_n + 4Log\left(\frac{1}{\varepsilon^2_{p_n}}\right) + O(1)\\ &\leq -p_n + 2Log\left(p_n|u_{p_n}|_\infty^{p_n-1}\right) + O(1)\\ &\leq C,\end{aligned}$$

where we have used (4.6).
We also have, for $x \in \partial\Omega_n$

$$-\Delta\psi_n(x) = -\Delta Z_n(x) + \Delta Z(x) = \Delta Z(x) = -\frac{16\sqrt{6}+|x|^2}{24(1+\frac{|x|^2}{8\sqrt{6}})^2} < 0.$$

-if $x \in \partial B(0,R_1)$, by Theorem 1.2 we deduce that

$$\psi_n(x) \leq c \quad \text{and} \quad -\Delta\psi_n(x) \leq c.$$

Now let us write down the equation satisfied by ψ_n.

$$\Delta^2\psi_n \leq \left(1+\frac{Z_n}{p_n}\right)^{p_n} - \left(1+\frac{Z}{p_n}\right)^{p_n}.$$

Using the convexity of $F(s) = \left(1+\frac{s}{p}\right)^p$ for $p>1$, we derive that

$$\Delta^2\psi_n \leq \left(1+\frac{Z_n}{p_n}\right)^{p_n-1}\psi_n = \frac{u_{p_n}^{p_n-1}(\varepsilon_{p_n}x + x_{p_n})}{|u_{p_n}|_\infty^{p_n-1}}\psi_n.$$

Since the maximum principle holds in $\Omega_n \diagdown B(0,R_1)$, we deduce that
$\psi_n \leq C$ in $\Omega_n \diagdown B(0,R_1)$ and it gives (4.7).
Now from (4.7) a contradiction follows easily. Indeed, using Theorem 1.2 and Lebesgue Theorem we derive that

$$\begin{aligned}p_n\int_\Omega u_{p_n}^{p_n+1} &= p_n|u_{p_n}|_\infty^{p_n+1}\varepsilon_{p_n}^4\int_{\Omega_n}\left(1+\frac{Z_n}{p_n}\right)^{p_n+1}\\ &= |u_{p_n}|_\infty^2\int_{\Omega_n}\left(1+\frac{Z_n}{p_n}\right)^{p_n+1}\\ &= \lim_{n\to+\infty}|u_{p_n}|_\infty^2\int_{\mathbb{R}^4}e^Z + o(1).\end{aligned}$$

Thus

$$\rho_4 e + o(1) = \lim_{n\to+\infty} |u_{p_n}|_\infty^2 \rho_4 + o(1)$$

which proves that $e = \lim_{n\to+\infty} |u_{p_n}|_\infty^2$, a contradiction with (4.6). Hence the result. □

5 Proof of Theorem 1.3

Let us start by proving the following crucial lemma:

Lemma 5.1 *We have that*

$$\lim_{p\to+\infty} p\int_\Omega u_p^p = \rho_4\sqrt{e}.$$

Proof. We have

$$p\int_\Omega u_p^{p+1} \le p|u_p|_\infty \int_\Omega u_p^p.$$

Thus, using Theorem 1.2 and Corollary 2.4, we obtain

$$\liminf_{p\to+\infty} p\int_\Omega u_p^p \ge \rho_4\sqrt{e}. \tag{5.1}$$

Arguing by contradiction, we suppose that there exists a sequence p_n such that

$$\lim_{n\to+\infty} p_n\int_\Omega u_{p_n}^{p_n} = \bar{C} > \rho_4\sqrt{e}.$$

We observe that

$$p_n\int_\Omega u_{p_n}^{p_n} = |u_{p_n}|_\infty \int_{\Omega_{p_n}} \left(1+\frac{Z_{p_n}}{p_n}\right)^{p_n}.$$

For R a large positive constant, we have

$$|u_{p_n}|_\infty \int_{B(0,R)} \left(1+\frac{Z_{p_n}}{p_n}\right)^{p_n} = |u_{p_n}|_\infty \int_{B(0,R)} e^Z + o_n(1) = \rho_4\sqrt{e} + o_n(1) + o_R(1).$$

Thus

$$\begin{aligned}\int_{\Omega_{p_n}\setminus B(0,R)} \left(1+\frac{Z_{p_n}}{p_n}\right)^{p_n} &= \frac{p_n}{|u_{p_n}|_\infty}\int_\Omega u_{p_n}^{p_n} - \int_{B(0,R)}\left(1+\frac{Z_{p_n}}{p_n}\right)^{p_n} \\ &= \frac{1}{\sqrt{e}}\left(\bar{C}-\rho_4\sqrt{e}\right) + o(1)\end{aligned}$$

and therefore

$$\lim_{n\to+\infty}\int_{\Omega_{p_n}\setminus B(0,R)} \left(1+\frac{Z_{p_n}}{p_n}\right)^{p_n} \ge C' > 0. \tag{5.2}$$

Now let us show (5.2) is not possible. Indeed, since by Corollary 2.4 we have

$$\rho_4 e + o(1) = p_n\int_\Omega u_{p_n}^{p_n+1} = |u_{p_n}|_\infty^2 \int_{\Omega_{p_n}} \left(1+\frac{Z_{p_n}}{p_n}\right)^{p_n+1}.$$

But, using Theorems 1.1 and 1.2, we derive that

$$|u_{p_n}|_\infty^2 \int_{B(0,R)} \left(1+\frac{Z_{p_n}}{p_n}\right)^{p_n+1} = e\int_{B(0,R)} e^Z + o_n(1) = \rho_4 e + o_n(1) + o_R(1).$$

Thus

$$\int_{\Omega_{p_n}\setminus B(0,R)} \left(1+\frac{Z_{p_n}}{p_n}\right)^{p_n+1} = o(1) \quad \text{for } R \text{ large.}$$

But, by Holder's inequality, we have

$$\int_{\Omega_{p_n}\setminus B(0,R)} \left(1+\frac{Z_{p_n}}{p_n}\right)^{p_n} \le |\Omega_{p_n}\setminus B(0,R)|^{\frac{1}{p_n+1}} \left(\int_{\Omega_{p_n}\setminus B(0,R)} \left(1+\frac{Z_{p_n}}{p_n}\right)^{p_n+1}\right)^{\frac{p_n}{p_n+1}}$$

$$\le \left(\frac{C}{\varepsilon_{p_n}^4}\right)^{\frac{1}{p_n+1}} o(1) = \left(Cp_n|u_{p_n}|_\infty^{p_n-1}\right)^{\frac{1}{p_n+1}} o(1) = o(1)$$

a contradiction with (5.2), then our lemma follows. □ Next, we state the following corollary which characterizes the set of blowup points of

$$v_p := pu_p.$$

Corollary 5.2 *Let* Γ *be the set defined by* (2.8)*. Then we have*

$$\Gamma = \{x_0\}, \quad \textit{where} \quad x_0 = \lim_{p\to+\infty} x_p.$$

Proof. Observe that

$$\forall\beta>0 \quad \mu_p\int_{B(x_0,\beta)} v_p^p = p\int_{B(x_0,\beta)} u_p^p = \|\, u_p\,\|_\infty \int_{B(0,\frac{\beta}{\varepsilon_p})} \left(1+\frac{Z_p}{p}\right)^p = \rho_4\sqrt{e}+o(1). \tag{5.3}$$

From (5.3) and Lemma 5.1, we easily derive our corollary. □ Now we are going to prove Theorem 1.3.

Proof of Theorem 1.3 First of all we show that

$$pu_p(x) \to \rho_4\sqrt{e}G(x,x_0) \quad \text{and} \quad \Delta(pu_p)(x) \to \rho_4\sqrt{e}\Delta G(x,x_0) \tag{5.4}$$

in the sense of distribution. For $\varphi\in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
\int_\Omega \Delta^2(pu_p)\varphi &= p\int_\Omega u_p^p\varphi \\
&= p\int_{\Omega_p} u_p^p(\varepsilon_p x+x_p)\varphi(\varepsilon_p x+x_p)\varepsilon_p^4 dx \\
&= |u_p|_\infty \int_{\Omega_p}\left(1+\frac{Z_p(x)}{p}\right)^p \varphi(\varepsilon_p x+x_p)dx \\
&= \sqrt{e}\int_{B(0,R)} e^Z\varphi(x_0) + o_p(1) + |u_p|_\infty\int_{\Omega_p\setminus B(0,R)}\left(1+\frac{Z_p(x)}{p}\right)^p\varphi(\varepsilon_p x+x_p)dx \\
&= \rho_4\sqrt{e}\varphi(x_0) + o_R(1) + o_p(1) + |u_p|_\infty\int_{\Omega_p\setminus B(0,R)}\left(1+\frac{Z_p(x)}{p}\right)^p\varphi(\varepsilon_p x+x_p)dx.
\end{aligned}$$