$6.95

Cat. No. TAS-1

PRINCIPLES AND APPLICATIONS OF TENSOR ANALYSIS

By MATTHEW S. SMITH

Principles and Applications of Tensor Analysis presents a detailed step-by-step development of tensor notation and theory, advanced concepts in tensor analysis, differential geometry, and analytical mechanics in tensor form.

CHAPTER 1. BASIC TENSOR THEORY

This chapter emphasizes the important concepts of relative tensors, covariant tensors, contravariant tensors, mixed tensors, metric tensors, base vectors, and Kronecker deltas.

CHARTER 2. CHRISTOFFEL SYMBOLS AND THE COVARIANT DERIVATIVES

Includes the Christoffel symbols, covariant derivatives, intrinsic derivatives, and Laplace's equation. Because of the many important applications, Laplace's equation is included in tensor form, using generalized curvilinear coordinates. Some of the more important applications in potential theory are included as illustrative examples.

$6.95

Cat. No. TAS-1

Principles & Applications of Tensor Analysis

by Matthew S. Smith

HOWARD W. SAMS & CO., INC.

FIRST EDITION

FIRST PRINTING—MARCH, 1963

PRINCIPLES & APPLICATIONS OF TENSOR ANALYSIS

Library of Congress Catalog Card Number: 63-11937

Preface

Since the extensive use of tensors by Einstein, important applications in other fields, such as differential geometry, classical mechanics, and the theory of elasticity, have evolved. Therefore, the ability to understand and apply tensors is a definite advantage to engineers, physicists, and applied mathematicians. Tensor equations are powerful analysis tools that give added insight to the understanding of the fundamental laws of physics and engineering.

This text, using practical examples throughout, systematically explains and demonstrates basic tensor theory and its applications. The book is divided into four chapters. The first two chapters present basic tensor theory, including Christoffel symbols, covariant derivatives, and Laplace's equation. The third chapter includes the Riemann-Christoffel tensors, and the application of tensor analysis to several topics in differential geometry. The latter were selected because they have numerous important applications in the general theory of relativity, analytical dynamics, and the theory of elasticity.

The fourth chapter presents the application of tensor analysis to some of the most important concepts in classical and relativistic mechanics. The section on classical dynamics includes Lagrange's equations of motion and a solution of the two-body problems. The equations for the two-body problem are written as a geodesic to give the student a feeling for dynamical trajectories treated as geodesics prior to the study of the general theory of relativity.

The sections on relativistic mechanics include a discussion of space-time for the special theory of relativity, the Lorentz transformations, and the relativistic equations for momentum, energy, and force. The text ends with a short discussion of curved space and Einstein's gravitational equations for the general theory of relativity.

It is hoped that an understanding of the material in this text will provide the student with the ability to solve a variety of problems using tensor analysis, as well as giving him an additional understanding of the related sciences.

MATTHEW S. SMITH

February, 1963

Contents

Chapter I

Basic Tensor Theory

Chapter I

Basic Tensor Theory

Tensor analysis is the study of invariant objects, whose properties must be independent of the co-ordinate systems used to describe the objects. A tensor is represented by a set of functions called components. For an object to be a tensor it must be an invariant that transforms from one acceptable co-ordinate system to another by the tensor laws. These laws are explained in detail, with examples, in Chapter 1.

Several examples of tensors are velocity vectors, base vectors, metric coefficients for the length of a line, Gaussian curvature, and the Newtonian gravitation potential.

Many of the important differential equations for physics, engineering, and applied mathematics can also be written as tensors. Examples of differential equations that can be written in tensor form are Lagrange's equations of motion and Laplace's equation. When an equation is written in tensor form it is in a general form that applies to all admissible co-ordinate systems.

SUMMATION NOTATION

The summation notation used throughout this text will be demonstrated explicitly by the examples in the first chapter. The general summation will be of the type:

$$S = a_1x_1 + a_2x_2 + \ldots + a_nx_n \qquad (1.0)$$

This type of summation is generally expressed in the calculus:

$$S = \sum_{i=1}^{n} a_i x^i \qquad (1.1)$$

The superscripts on x are not powers; they are used to distinguish between the various x's. In rectangular cartesian co-ordinates and vector notation, Equation 1.1 would be:

$$S = \sum_{i=1}^{3} a_i x^i$$

where,

$x^1 = x, x^2 = y, x^3 = z$

$a_1 = i, a_2 = j, a_3 = k$

With this interpretation of Equation 1.1 and the specific values for a_i and x^i as noted, sum S would be:

$$S = ix + jy + kz$$

For additional simplification, Einstein dropped the $\sum$ in Equation 1.1 and the summation is then expressed:

$$S = a_ix^i \qquad (1.2)$$

In conclusion, it is to be remembered that Equations 1.0, 1.1, and 1.2 are equivalent, and x^i expresses variables, not powers of x.

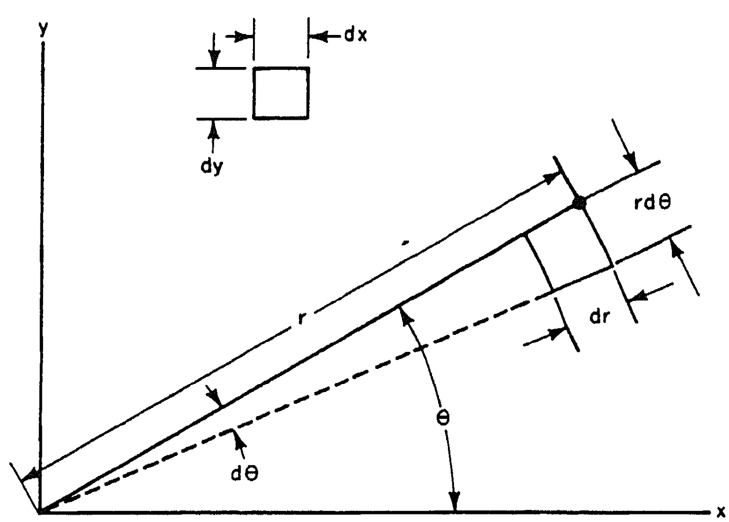
In the subsequent portions of the text, a superscript index will indicate a contravariant tensor, while a subscript index will indicate a covariant tensor.

The rank of a tensor is the sum of the covariant and contravariant indexes. This will be explained in detail in the section of this chapter on higher rank and mixed tensors.

RELATIVE TENSORS

The term relative tensor is used to describe scalars that are transformed from one co-ordinate system to another by means of the functional determinate known as the Jacobian. To illustrate this concept, the differential increment of area (dA) is indicated in Fig. 1-1.

In cartesian co-ordinates (x, y) it is:



$$dA = dx\,dy$$

In polar co-ordinates (r, Θ) it is:

$$dA = r\,d\Theta\,dr$$

Now, the connection between the x, y cartesian co-ordinates and the r, Θ polar co-ordinates is:

$$x = r \cos \Theta$$

$$y = r \sin \Theta$$

$$r = (x^2 + y^2)^{1/2}$$

$$\Theta = \tan^{-1} \frac{y}{x}$$

The Jacobian of the cartesian co-ordinates with respect to the polar co-ordinates is formed from the following partial derivatives:

$$\frac{\partial x}{\partial r} = \cos \Theta, \quad \frac{\partial x}{\partial \Theta} = -r \sin \Theta$$

$$\frac{\partial y}{\partial r} = \sin \Theta, \quad \frac{\partial y}{\partial \Theta} = r \cos \Theta$$

This set of partial derivatives are used to form the following Jacobian:

$$\begin{vmatrix} \cos \Theta & -r \sin \Theta \\ \sin \Theta & r \cos \Theta \end{vmatrix} = r\,(\cos^2\Theta + \sin^2\Theta) = r \qquad (1.3)$$

with respect to the cartesian co-ordinates is formed from the following partial derivatives:

$$\frac{\partial r}{\partial x} = \cos \Theta, \quad \frac{\partial r}{\partial y} = \sin \Theta$$

$$\frac{\partial \Theta}{\partial x} = -\frac{1}{r}\sin \Theta, \quad \frac{\partial \Theta}{\partial y} = \frac{1}{r}\cos \Theta$$

This set of partial derivatives are used to form the following Jacobian:

$$\begin{vmatrix} \cos \Theta & \sin \Theta \\ -\frac{1}{r}\sin \Theta & \frac{1}{r}\cos \Theta \end{vmatrix} = \frac{1}{r}\,(\cos^2\Theta + \sin^2\Theta) = \frac{1}{r} \qquad (1.4)$$

Now, returning to the expression for differential area in cartesian co-ordinates and polar co-ordinates, the following equation can be written:

$$S\,dx\,dy = \overline{S}\,dr\,d\Theta$$

where,

$$S = 1$$

$$\overline{S} = r$$

S and $\overline{S}$ are called relative tensors, as they are related by the equations:

$$\left| \partial x^i \right|^n$$

$$\bar{S} = \left| \frac{\partial x^i}{\partial y^j} \right|^n S \qquad (1.6)$$

Exponent n in Equations 1.5 and 1.6 is used to determine the weight of a relative scalar. The examples in this section are relative scalars having a weight equaling one; therefore, $n = 1$. An absolute scalar has a weight of zero; i.e., $n = 0$.

To illustrate Equation 1.5, we use the values:

$$\bar{S} = r$$

$$\left| \frac{\partial y^i}{\partial x^j} \right| = \begin{vmatrix} \cos \Theta & \sin \Theta \\ -\frac{1}{r} \sin \Theta & \frac{1}{r} \cos \Theta \end{vmatrix} = \frac{1}{r}$$

y^i ranges from $i = 1$ to $i = 2$

x^j ranges from $j = 1$ to $j = 2$

$$y^1 = r, \quad x^1 = x$$

$$y^2 = \Theta, \quad x^2 = y$$

$$S = \frac{1}{r}(r) = 1 \qquad (1.7)$$

Equation 1.7 is the desired result.

Now the notion of relative tensors can be extended to volumes and mass. To illustrate this concept, we start with the equation for an incremental mass in orthogonal cartesian coordinates.

$$dM = \rho \, dx \, dy \, dz \qquad (1.8)$$

Now the incremental mass in spherical co-ordinates is written in terms of relative tensor $\bar{S}$:

$$d\overline{M} = \overline{S}\,dr\,d\Phi\,d\Theta$$

$\overline{S}$ is evaluated by the relative tensor equation:

$$\overline{S} = \left|\frac{\partial x^i}{\partial y^j}\right| S \tag{1.9}$$

In this example S = ρ, where ρ is called the scalar density.

$$x^1 = x,\ x^2 = y,\ x^3 = z$$

$$y^1 = r,\ y^2 = \Phi,\ y^3 = \Theta$$

The geometrical relationship between the cartesian co-ordinates and the spherical co-ordinates is indicated in Fig. 1-2. The corresponding mathematical relationship between the co-ordinates is:

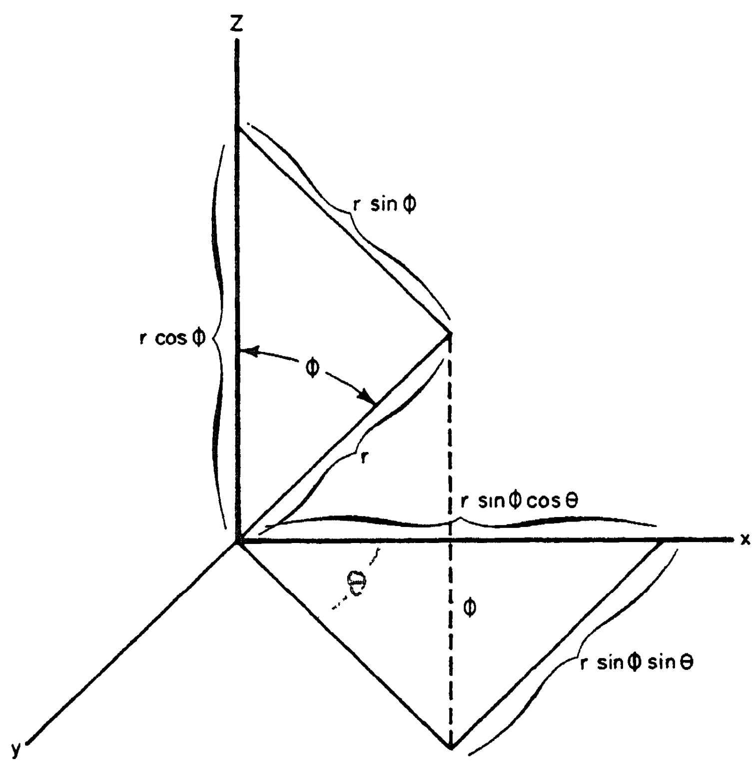


Fig. 1-2.

$$\left.\begin{aligned} x &= r \sin\Phi \cos\Theta \\ y &= r \sin\Phi \sin\Theta \\ z &= r \cos\Phi \end{aligned}\right\} \qquad (1.10)$$

The partial derivatives for the Jacobian $\left|\frac{\partial x^i}{\partial y^j}\right|$ are:

$$\frac{\partial x}{\partial r} = \sin\Phi \cos\Theta, \frac{\partial x}{\partial \Phi} = r \cos\Phi \cos\Theta, \frac{\partial x}{\partial \Theta} =$$

$$-r \sin\Phi \sin\Theta$$

$$\frac{\partial y}{\partial r} = \sin\Phi \sin\Theta, \frac{\partial y}{\partial \Phi} = r \cos\Phi \sin\Theta, \frac{\partial y}{\partial \Theta} =$$

$$r \sin\Phi \cos\Theta$$

$$\frac{\partial z}{\partial r} = \cos\Phi, \frac{\partial z}{\partial \Phi} = -r \sin\Phi, \frac{\partial z}{\partial \Theta} = 0$$

Using these values, the resultant determinate is:

$$\begin{vmatrix} \sin\Phi \cos\Theta & r \cos\Phi \cos\Theta & -r \sin\Phi \sin\Theta \\ \sin\Phi \sin\Theta & r \cos\Phi \sin\Theta & r \sin\Phi \cos\Theta \\ \cos\Phi & -r \sin\Phi & 0 \end{vmatrix} = r^2 \sin\Phi \qquad (1.11)$$

Now, Equation 1.6 can be evaluated:

$$\overline{S} = r^2 \sin\Phi \, \rho \qquad (1.12)$$

and the equation for $d\overline{M}$ is:

$$d\overline{M} = \rho\, r^2 \sin \Phi \, dr \, d\Phi \, d\Theta \qquad (1.13)$$

Equation 1.13 is the desired result for $d\overline{M}$. If the value for $d\overline{M}$ had been given initially in spherical co-ordinates, the corresponding value in cartesian co-ordinates could be found by the equation:

$$dM = S \, dx \, dy \, dz \qquad (1.14)$$

where,

$$S = \left| \frac{\partial y^i}{\partial x^j} \right| \overline{S} \qquad (1.5)$$

ADMISSIBLE TRANSFORMATIONS

From the previous examples, it has been demonstrated that relative tensors transform from one co-ordinate system to another by means of the functional determinate known as the Jacobian. Since a relative tensor is defined to be a function of the Jacobian, a necessary and sufficient condition for an admissible transformation of co-ordinates is that it is a member of a set in which the Jacobian does not vanish.

This condition is also necessary and sufficient for absolute tensors. Therefore, the set of all admissible transformations of co-ordinates form a group with nonvanishing Jacobians. If notation $|J|$ is used for the Jacobian, the definition for an admissible transformation of co-ordinates can be expressed:

$$|J| \neq 0 \qquad (1.15)$$

Another property of an admissible transformation is:

$$\left| \frac{\partial x^i}{\partial y^j} \right| \left| \frac{\partial y^i}{\partial x^j} \right| = 1 \qquad (1.16)$$

An example of Equation 1.16 can be found in Jacobian Equations 1.3 and 1.4.

$$(r)\left(\frac{1}{r}\right) = 1 \qquad (1.17)$$

N DIMENSIONAL SPACE

In general terms a co-ordinate system represents a one-to-one correspondence of a point or object with a set of numbers. To measure distance, we can use a rectangular cartesian co-ordinate system. This is called a metric manifold, or space of V_3.

Now, a space or manifold of N dimensions is expressed by the symbol V_N; and it is a co-ordinate system of N dimensions, if for each set of N numbers there is one corresponding point or object.

A sub space V_M where $M = N - 1$ is called a hypersurface. An example of hypersurface in Euclidean space is a plane. It is a hypersurface of V_2.

CONTRAVARIANT TENSORS

The prototype for contravariant tensors is the vector formed by taking the total differential of a variable in one co-ordinate system with respect to the variables in another admissible co-ordinate system. It is a contravariant tensor Rank 1. To present the concept, cartesian co-ordinates x and y; and polar co-ordinates r and Θ as indicated in Fig. 1-1 are used.

$$\left.\begin{aligned} x &= r \cos \Theta \\ y &= r \sin \Theta \end{aligned}\right\} \qquad (1.18)$$

Differential dx in terms of r and Θ is a contravariant tensor Rank 1.