Reviel Netz

The Works of Archimedes

Translation and Commentary

Volume I: The Two Books On the Sphere and the Cylinder

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The Works of Archimedes

Archimedes was the greatest scientist of antiquity and one of the greatest of all time. This book is Volume I of the first fully fledged translation of his works into English. It is also the first publication of a major ancient Greek mathematician to include a critical edition of the diagrams, and the first translation into English of Eutocius' ancient commentary on Archimedes. Furthermore, it is the first work to offer recent evidence based on the Archimedes Palimpsest, the major source for Archimedes, lost between 1915 and 1998. A commentary on the translated text studies the cognitive practice assumed in writing and reading the work, and it is Reviel Netz's aim to recover the original function of the text as an act of communication. Particular attention is paid to the aesthetic dimension of Archimedes' writings. Taken as a whole, the commentary offers a groundbreaking approach to the study of mathematical texts.

REVIEL NETZ is Associate Professor of Classics at Stanford University. His first book, *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History* (1999), was a joint winner of the Runciman Award for 2000. He has also published many scholarly articles, especially in the history of ancient science, and a volume of Hebrew poetry, *Adayin Bahuc* (1999). He is currently editing The Archimedes Palimpsest and has another book forthcoming with Cambridge University Press, *From Problems to Equations: A Study in the Transformation of Early Mediterranean Mathematics.*

THE WORKS OF
ARCHIMEDES

Translated into English, together with
Eutocius' commentaries, with commentary,
and critical edition of the diagrams

REVIEL NETZ
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Volume I
The Two Books *On the Sphere and the Cylinder*

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To Maya

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Perhaps the greatest pleasure in working on this book was the study of the manuscripts of Archimedes kept in several libraries: the National Library in Paris, the Marcian Library in Venice, the Laurentian Library in Florence and the Vatican Library in Rome. The librarians at these institutions were all very kind and patient (not easy, when your reader bends over diagrams, ruler and compass in hand!). I wish to thank them all for their help.

Special words of thanks go to the Walters Art Museum in Baltimore, where the Archimedes Palimpsest has recently been entrusted for conservation. I am deeply grateful to the Curator of Manuscripts there, William Noel, to the conservator of manuscripts, Abigail Quandt, to the imagers of the manuscript, especially Bill Christens-Barry, Roger Easton, and Keith Knox and finally, and most importantly, to the anonymous owner of the manuscript, for allowing study of this unique document.

My most emphatic words of thanks, perhaps, should go to Cambridge University Press, for undertaking this complicated project, and for patience when, with the Archimedes Palimpsest rediscovered, delay – of the most welcome kind – was suddenly imposed upon us. I thank Pauline Hire, the Classics Editor in the Press at the time this work was begun, and Michael Sharp, current Classics Editor, for invaluable advice, criticism and friendliness. Special words of thanks go to my student, Alexander Lee, for his help in proofreading the manuscript.

To mention by name all those whose kind words and good advice have sustained this study would amount to a publication of my private list of addresses. Let it be said instead that this work is a product of many intersecting research communities – in the History of Greek Mathematics, in Classics, in the History and Philosophy of Science, as well as in other fields – from whom I continue to learn, and for whom I have produced this work, as a contribution to an ongoing common study – and as a token of my gratitude.

INTRODUCTION

I GOAL OF THE TRANSLATION

The extraordinary influence of Archimedes over the scientific revolution was due in the main to Latin and Greek–Latin versions handwritten and then printed from the thirteenth to the seventeenth centuries.[1] Translations into modern European languages came later, some languages served better than others. There are, for instance, three useful French translations of the works of Archimedes,[2] of which the most recent, by C. Mugler – based on the best text known to the twentieth century – is still easily available. A strange turn of events prevented the English language from possessing until now any full-blown translation of Archimedes. As explained by T. L. Heath in his important book, *The Works of Archimedes*, he had set out there to make Archimedes accessible to contemporary mathematicians to whom – so he had thought – the mathematical contents of Archimedes' works might still be of practical (rather than historical) interest. He therefore produced a paraphrase of the Archimedean text, using modern symbolism, introducing consistency where the original is full of tensions, amplifying where the text is brief, abbreviating where it is verbose, clarifying where it is ambiguous: almost as if he was preparing an undergraduate textbook of "Archimedean Mathematics." All this was done in good faith, with Heath signalling his practices very clearly, so that the book is still greatly useful as a mathematical gloss to Archimedes. (For such a mathematical gloss, however, the best work is likely to remain Dijksterhuis' masterpiece from 1938 (1987), *Archimedes*.) As it turned out, Heath had acquired in the twentieth century a special position in the English-speaking world. Thanks to his good English style, his careful and highly scholarly translation of Euclid's *Elements*, and, most important, thanks to the sheer volume of his activity, his works acquired the reputation of finality. Such reputations are always

[1] See in particular Clagett (1964–84), Rose (1974), Hoyrup (1994).
[2] Peyrard (1807), Ver Eecke (1921), Mugler (1970–74).

deceptive, nor would I assume the volumes, of which you now hold the first, are more than another transient tool, made for its time. Still, you now hold the first translation of the works of Archimedes into English.

The very text of Archimedes, even aside from its translation, has undergone strange fortunes. I shall return below to describe this question in somewhat greater detail, but let us note briefly the basic circumstances. None of the three major medieval sources for the writings of Archimedes survives intact. Using Renaissance copies made only of one of those medieval sources, the great Danish scholar J. L. Heiberg produced the first important edition of Archimedes in the years 1880–81 (he was twenty-six at the time the first volume appeared). In quick succession thereafter – a warning to all graduate students – two major sources were then discovered. The first was a thirteenth-century translation into Latin, made by William of Moerbeke, found in Rome and described in 1884,[3] and then, in 1906, a tenth-century Palimpsest was discovered in Istanbul.[4] This was a fabulous find indeed, a remarkably important text of Archimedes – albeit rewritten and covered in the thirteenth century by a prayer book (which is why this manuscript is now known as a *Palimpsest*). Moerbeke's translation provided a much better text for the treatise *On Floating Bodies*, and allowed some corrections on the other remaining works; the Palimpsest offered a better text still for *On Floating Bodies* – in Greek, this time – provided the bulk of a totally new treatise, the *Method*, and a fragment of another, the *Stomachion*. Heiberg went on to provide a new edition (1910–15) reading the Palimpsest as best he could. We imagine him, through the years 1906 to 1915, poring in Copenhagen over black-and-white photographs, the magnifying glass at hand – a Sherlock Holmes on the Sound. A fine detective work he did, deciphering much (though, now we know, far from all) of Archimedes' text. Indeed, one wishes it was Holmes himself on the case; for the Palimpsest was meanwhile gone, Heiberg probably never even realizing this. Rumored to be in private hands in Paris yet considered effectively lost for most of the twentieth century, the manuscript suddenly reappeared in 1998, considerably damaged, in a sale at New York, where it fetched the price of two million dollars. At the time of writing, the mystery of its disappearance is still far from being solved. The manuscript is now being edited in full, for the first time, using modern imaging techniques. Information from this new edition is incorporated into this translation. (It should be noted, incidentally, that Heath's version was based solely on Heiberg's first edition of Archimedes, badly dated already in the twentieth century.) Work on this first volume of translation had started even before the Palimpsest resurfaced. Fortunately, a work was chosen – the books *On the Sphere and the Cylinder*, together with Eutocius' ancient commentary – that is largely independent from the Palimpsest. (Eutocius is not represented in the Palimpsest, while Archimedes' text of this work is largely unaffected by the readings of the Palimpsest.) Thus I can move on to publishing this volume even before the complete re-edition of the Palimpsest has been made, basing myself on Heiberg's edition together with a partial consultation of the Palimpsest. The

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3 Rose (1884). 4 Heiberg (1907).

translations of *On Floating Bodies*, the *Method* and the *Stomachion* will be published in later volumes, when the Palimpsest has been fully deciphered. It is already clear that the new version shall be fundamentally different from the one currently available.

The need for a faithful, complete translation of Archimedes into English, based on the best sources, is obvious. Archimedes was not only an outstanding mathematician and scientist (clearly the greatest of antiquity) but also a very influential one. Throughout antiquity and the middle ages, down to the scientific revolution and even beyond, Archimedes was a living presence for practicing scientists, an authority to emulate and a presence to compete with. While several distinguished studies of Archimedes had appeared in the English language, he can still be said to be the least studied of the truly great scientists. Clearly, the history of science requires a reliable translation that may serve as basis for scholarly comment. This is the basic purpose of this new translation.

There are many possible barriers to the reading of a text written in a foreign language, and the purpose of a scholarly translation as I understand it is to remove all barriers having to do with the foreign language itself, leaving all other barriers intact. The Archimedean text approaches mathematics in a way radically different from ours. To take a central example, this text does not use algebraic symbolism of any kind, relying, instead, upon a certain formulaic use of language. To get habituated to this use of language is a necessary part of understanding how Archimedes thought and wrote. I thus offer the most faithful translation possible. Differences between Greek and English make it impossible, of course, to provide a strict one-to-one translation (where each Greek word gets translated constantly by the same English word) and thus the translation, while faithful, is not literal. It aims, however, at something close to literality, and, in some important intersections, the English had to give way to the Greek. This is not only to make sure that specialist scholars will not be misled, but also because whoever wishes to read Archimedes, should be able to read *Archimedes*. Style and mode of presentation are not incidental to a mathematical proof: they constitute its soul, and it is this soul that I try, to the best of my ability, to bring back to life.

The text resulting from such a faithful translation is difficult. I therefore surround it with several layers of interpretation.

- I intervene in the body of the text, in clearly marked ways. Glosses added within the standard pointed-brackets notation (<. . .>) are inserted wherever required, the steps of proofs are distinguished and numbered, etc. I give below a list of all such conventions of intervention in the text. The aim of such interventions is to make it easier to construe the text as a sequence of meaningful assertions, correctly parsing the logical structure of these assertions.
- Footnotes add a brief and elementary mathematical commentary, explaining the grounds for the particular claims made. Often, these take the form of references to the tool-box of known results used by Archimedes. Sometimes, I refer to Eutocius' commentary to Archimedes (see below). The aim of these footnotes, then, is to help the readers in checking the validity of the argument.

- A two-part set of comments follows each proposition (or, in some cases, units of text other than propositions):
  - The first are *textual comments*. Generally speaking, I follow Heiberg's (1910–15) edition, which seems to remain nearly unchanged, for the books *On the Sphere and the Cylinder*, even with the new readings of the Palimpsest. In some cases I deviate from Heiberg's text, and such deviations (excepting some trivial cases) are argued for in the textual comments. In other cases – which are very common – I follow Heiberg's text, while doubting Heiberg's judgment concerning the following question. Which parts of the text are genuine and which are interpolated? Heiberg marked what he considered interpolated, by square brackets ([. . .]). I print Heiberg's square brackets in my translation, but I very often question them within the textual comments.
  - The second are *general comments*. My purpose there is to develop an interpretation of certain features of Archimedes' writing. The comments have the character not of a reference work, but of a monograph. This translation differs from other versions in its close proximity to the original; it maps, as it were, a space very near the original writing. It is on this space that I tend to focus in my general comments. Thus I choose to say relatively little on wider mathematical issues (which could be equally accessed through a very distant translation), only briefly supply biographical and bibliographical discussions, and often focus instead on narrower cognitive or even linguistic issues. I offer three apologies for this choice. First, such comments on cognitive and linguistic detail are frequently necessary for understanding the basic meaning of the text. Second, I believe such details offer, taken as a whole, a central perspective on Greek mathematical practices in general, as well as on Archimedes' individual character as an author. Third and most important, having now read many comments made in the past by earlier authors, I can no longer see such comments as "definitive." Mine are "comments," not "commentary," and I choose to concentrate on what I perceive to be of relevance to contemporary scholarship, based on my own interest and expertise. Other comments, of many different kinds, will certainly be made by future readers of Archimedes. Readers interested in more mathematical commentary should use Eutocius as well as Dijksterhuis (1987), those interested in more biographical and historical detail on the mathematicians mentioned should use Knorr (1986), (1989), and those looking for more bibliographic references should use Knorr (1987) (which remains, sixteen years later nearly complete). (Indeed, as mentioned above, Archimedes is not intensively studied.)
- Following the translation of Archimedes' work, I add a translation of Eutocius' commentary to Archimedes. This is a competent commentary and the only one of its kind to survive from antiquity. Often, it offers a very useful commentary on the mathematical detail, and in many cases it has unique historical significance for Archimedes and for Greek mathematics in general. The translation of Eutocius follows the conventions of the translation of Archimedes, but I do not add comments to his text, instead supplying, where necessary, fuller footnotes.

A special feature of this work is a critical edition of the diagrams. Instead of drawing my own diagrams to fit the text, I produce a reconstruction based on the independent extant manuscripts, adding a critical apparatus with the variations between the manuscripts. As I have argued elsewhere (Netz [forthcoming]), I believe that this reconstruction may represent the diagrams as available in late antiquity and, possibly, at least in some cases, as produced by Archimedes himself. Thus they offer another, vital clue to our main question, how Archimedes thought and wrote. I shall return below to explain briefly the purpose and practices of this critical edition.

Before the translation itself I now add a few brief preliminary notes.

2 PRELIMINARY NOTES: CONVENTIONS

2.1 Some special conventions of Greek mathematics

In the following I note certain practices to be found in Archimedes' text that a modern reader might find, at first, confusing.

1. Greek word order is much freer than English word order and so, selecting from among the wider set of options, Greek authors can choose one word order over another to emphasize a certain idea. Thus, for instance, instead of writing "A is equal to B," Greek authors might write "to B is equal A." This would stress that the main information concerns B, not A – word order would make B, not A, the focus. (For instance, we may have been told something about B, and now we are being told the extra property of B, that it is equal to A.) Generally speaking, such word order cannot be kept in the English, but I try to note it when it is of special significance, usually in a footnote.

2. The summation of objects is often done in Greek through ordinary conjunction. Thus "the squares ΑΒΓΔ and ΕΖΗΘ" will often stand for what we may call "the square ΑΒΓΔ plus the square ΕΖΗΘ." As an extension of this, the ordinary plural form can serve, as well, to represent summation: "the squares ΑΒΓΔ, ΕΖΗΘ" (even without the "and" connector!) will then mean "the square ΑΒΓΔ plus the square ΕΖΗΘ." In such cases, the sense of the expression is in itself ambiguous (the following predicate may apply to the sum of the objects, or it may apply to each individually), but such expressions are generally speaking easily disambiguated in context. Note also that while such "implicit" summations are very frequent, summation is often more explicit and may be represented by connectors such as "together with," "taken together," or simply "with."

3. The main expression of Greek mathematics is that of proportion:

As A is to B, so is C to D.

(A, B, C, and D being some mathematical objects.) This expression is often represented symbolically, in modern texts, by

A:B::C:D

and I will use such symbolism in my footnotes and commentary. In the main text I shall translate, of course, the original non-symbolic form. Note especially that this expression may become even more concise, e.g.:

As A is to B, C to D, As A to B, C to D.

And that it may have more complex syntax, especially:

A has to B the same ratio as C has to D, A has to B a greater ratio than C has to D.

The last example involves an obvious extension of proportion, to ratio-inequalities, i.e. A:B>C:D. More concisely, this may be expressed by:

A has to B a greater ratio than C to D.

4. Greek mathematical propositions have, in many cases, the following six parts:

- *Enunciation*, in which the claim of the proposition is made, in general terms, without reference to the diagram. It is important to note that, generally speaking, the enunciation is equivalent to a conditional statement that *if* x is the case, *then* so is y.
- *Setting-out*, in which the antecedent of the claim is re-stated, in particular terms referring to the diagram (with the example above, x is re-stated in particular reference to the diagram).
- *Definition of goal*, in which the consequent of the claim is re-stated, as an exhortation addressed by the author to himself: "I say that . . . ," "it is required to prove that . . . ," again in the particular terms of the diagram (with the same example, we can say that y is re-stated in particular reference to the diagram).
- *Construction*, in which added mathematical objects (beyond those required by the setting-out) may be introduced.
- *Proof*, in which the particular claim is proved.
- *Conclusion*, in which the conclusion is reiterated for the general claim from the enunciation.

Some of these parts will be missing in most Archimedean propositions, but the scheme remains a useful analytical tool, and I shall use it as such in my commentary. The reader should be prepared in particular for the following difficulty. It is often very difficult to follow the enunciations as they are presented. Since they do not refer to the particular diagram, they use completely general terms, and since they aspire to great precision, they may have complex qualifications and combinations of terms. I wish to exonerate myself: this is not a problem of my translation, but of Greek mathematics. Most modern readers find that they can best understand such enunciations by reading, first, the setting-out and the definition of goal, with the aid of the diagram. Having read this, a better sense of the *dramatis personae* is gained, and the enunciation may be deciphered. In all probability the ancients did the same.

2.2 Special conventions adopted in this translation

1. The main "<. . .>" policy:

Greek mathematical proofs always refer to concrete objects, realized in the diagram. Because Greek has a definite article with a rich morphology, it can elide the reference to the objects, leaving the definite article alone. Thus the Greek may contain expressions such as

"The by the AB, BΓ"

whose reference is

"The <rectangle contained> by the <lines> AB, BΓ"

(the morphology of the word "the" determines, in the original Greek, the identity of the elided expressions, given of course the expectations created by the genre).

In this translation, most such elided expressions are usually added inside pointed brackets, so as to make it possible for the reader to appreciate the radical concision of the original formulation, and the concreteness of reference – while allowing me to represent the considerable variability of elision (very often, expressions have only partial elision). This variability, of course, will be seen in the fluctuating positions of pointed brackets:

"The <rectangle contained> by the <lines> AB, BΓ," as against, e.g.

"The <rectangle> contained by the <lines> AB, BΓ."

(Notice that I do not at all strive at consistency *inside* pointed brackets. Inside pointed brackets I put whatever seems to me, in context, most useful to the reader; the duties of consistency are limited to the translation proper, outside pointed brackets.)

The main exception to my general pointed-brackets policy concerns points and lines. These are so frequently referred to in the text that to insist, always, upon a strict representation of the original, with expressions such as

"The <point> **A**," "The <line> **AB**"

would be tedious, while serving little purpose. I thus usually write, simply:

A, AB

and, in the less common cases of a non-elliptic form:

"The point A," "The line AB"

The price paid for this is that (relatively rarely) it is necessary to stress that the objects in question are points or lines, and while the elliptic Greek expresses this through the definite article, my elliptic "A," "AB" does not. Hence I need to introduce, here and there, the expressions:

"The <point> **A**," "The <line> **AB**"

but notice that these stand for precisely the same as

A, AB.

2. The "<=. .>" sign is also used, in an obvious way, to mean essentially the same as the "[*Scilicet*. . . .]" abbreviation. Most often, the expression following the "=" will disambiguate pronouns which are ambiguous in the English (but which, in the Greek, were unambiguous thanks to their morphology).

3. Square brackets in the translation ("[. . .]") represent the square brackets in Heiberg's (1910–15) edition. They signify units of text which according to Heiberg were interpolated.

4. Two sequences of numbering appear inside standard brackets. The Latin alphabet sequence "(a) . . . (b) . . ." is used to mark the sequence of constructions: as each new item is added to the construction of the geometrical configuration (following the setting-out) I mark this with a letter in the sequence of the Latin alphabet. Similarly, the Arabic number sequence "(1) . . . (2) . . ." is used to mark the sequence of assertions made in the course of the proof: as each new assertion is made (what may be called "a step in the argument"), I mark this with a number. This is meant for ease of reference: the footnotes and the

commentary refer to constructions and to claims according to their letters or numbers. **Note that this is purely my editorial intervention, and that the original text had nothing corresponding to such brackets.** (The same is true for punctuation in general, for which see point 6 below.) Also note that these sequences refer only to construction and proof: enunciation, setting-out, and definition of goal are not marked in similar ways.

5. The "/. . ./" symbolism: for ease of reference, I find it useful to add in titles for elements of the text of Archimedes, whether general titles such as "introduction" or numbers referring to propositions. I suspect Archimedes' original text had neither, and such titles and numbers are therefore mere aids for the reader in navigating the text.

6. Ancient Greek texts were written without spacing or punctuation: they were simply a continuous stream of letters. Thus punctuation as used in modern editions reflects, at best, the judgments of late antiquity and the middle ages, more often the judgments of the modern editor. I thus use punctuation freely, as another editorial tool designed to help the reader, but in general I try to keep Heiberg's punctuation, in deference to his superb grasp of the Greek mathematical language, and in order to facilitate simultaneous use of my translation and Heiberg's edition.

7. Greek diagrams can be characterized as "qualitative" rather than "quantitative." This is very difficult to define precisely, and is best understood as a warning: **do not assume that relations of size in the diagram represent relations of size in the depicted geometrical objects**. Thus, two geometrical lines may be assumed equal, while their diagrammatic representation is of two unequal lines and, even more confusingly, two geometrical lines may be assumed *un*equal, while their diagrammatic representation is of two *equal* lines. Similar considerations apply to angles etc. What the diagram most clearly *does* represent are relations of connection between the geometrical constituents of the configuration (what might be loosely termed "topological properties"). Thus, in an extreme case, the diagram may concentrate on representing the fact that two lines touch at a single point, ignoring another geometrical fact, that one of the lines is straight while the other is curved. This happens in a series of propositions from 21 onwards, in which a dodecagon is represented by twelve *arcs*; but this is an extreme case, and generally the diagram may be relied upon for such basic qualitative distinctions as straight/curved. See the following note on purpose and practices of the critical edition of diagrams.

2.3 Purpose and practices of the critical edition of diagrams

The main purpose of the critical edition of diagrams is to reconstruct the earliest form of diagrams recoverable from the manuscript evidence. It should be stressed that the diagrams across the manuscript tradition are strikingly similar to each other, often in quite trivial detail, so that there is hardly a question that they derive from a common archetype. For most of the text translated here, diagrams are preserved only for one Byzantine tradition, that of codex A (see below, note on the text of Archimedes). However, for most of the diagrams from