SMF/AMS TEXTS *and* MONOGRAPHS

VOLUME 14

# An Initiation to Logarithmic Sobolev Inequalities

Gilles Royer

American Mathematical Society
Société Mathématique de France

# An Initiation to Logarithmic Sobolev Inequalities

SMF/AMS TEXTS *and* MONOGRAPHS • *Volume* 14

Cours Spécialisés • *Numéro* 5 • 1999

# An Initiation to Logarithmic Sobolev Inequalities

Gilles Royer

*Translated by*
Donald Babbitt

American Mathematical Society
Société Mathématique de France

Une Initiation aux Inégalités de Sobolev Logarithmiques
An Initiation to Logarithmic Sobolev Inequalities

Gilles Royer

Originally published in French by Société Mathématique de France.

Copyright © 1999 Société Mathématique de France

Translated from the French by Donald Babbitt

2000 *Mathematics Subject Classification.* Primary 60–02; Secondary 35J85, 47B25, 47D07, 60J60, 82C99.

For additional information and updates on this book, visit
**www.ams.org/bookpages/smfams-14**

**Library of Congress Cataloging-in-Publication Data**

Royer, Gilles.
[Initiation aux inégalités de Sobolev logarithmiques. English]
An initiation to logarithmic Sobolev inequalities / Gilles Royer ; translated by Donald Babbitt.
p. cm. — (SMF/AMS texts and monographs, ISSN 1525-2302 ; v. 14) (Cours spécialisés, ISSN 1284-6090 ; 5)
Includes bibliographical references.
ISBN-13: 978-0-8218-4401-4 (alk. paper)
ISBN-10: 0-8218-4401-6 (alk. paper)
1. Ergodic theory. 2. Logarithmic functions. 3. Semigroups of operators. 4. Differential inequalities. I. Title.

QA313.R6913 2007
515′.48—dc22 2007060798

**Copying and reprinting.** Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294, USA. Requests can also be made by e-mail to reprint-permission@ams.org.

© 2007 by the American Mathematical Society. All rights reserved.
The American Mathematical Society retains all rights
except those granted to the United States Government.
Printed in the United States of America.

The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at http://www.ams.org/

10 9 8 7 6 5 4 3 2 1 12 11 10 09 08 07

# Contents

# Preface

This book contains the material that was essentially covered in a course "de troisième cycle"[1] taught during the second semester of the 1996-1997 academic year at the University d'Orléans. The goal of this course was to give an exposition of an example of the use of logarithmic Sobolev inequalities coming primarily from two papers by B. Zegarlinski [**Zeg90, Zeg96**]. The example is concerned with real spin models with weak interactions on a lattice where one can apply a classic method due to Dobrushin; see notably [**Dob70**]. For these models, we give a proof of the uniqueness of the Gibbs measure by showing the exponential stabilization of the stochastic evolution of an infinite dimensional diffusion process which generalizes the case of the Glauber dynamics for the Ising model. Although these models are technically more complicated than the Ising model, one still uses familiar techniques, e.g., using Ito's stochastic integral calculus to construct and study diffusion processes, as well as utilizing the well-known properties of self-adjoint differential operators on $\mathbb{R}^n$ and Sobolev and Poincaré inequalities in their original setting. These models also utilize in a natural way some elegant results on logarithmic Sobolev inequalities such as the Bakry-Émery and Herbst inequalities. Interestingly, these models are simplifications of the Nelson models of Euclidean fields where Gross first introduced logarithmic Sobolev inequalities.[2]

In this book we introduce in a self-contained manner the basic notions of self-adjoint operators, diffusion processes, and Gibbs measures. The chapter on logarithmic Sobolev inequalities is enriched by adding applications to Markov chains so as not to remain in too special a setting. The reader will find indications of some recent applications of logarithmic Sobolev inequalities to statistical mechanics at the end of Chapter 5.

I would like to thank my colleagues S. Roelly and P. Maheux for very useful discussions as well as the students of the DEA d'Orléans, in particular, G. Salin.

**Note added to the original Preface.** The translation presented here differs from the French original by a small number of corrections. Since the original course was given, logarithmic Sobolev inequalities have been the subject of many articles. We recommend that interested readers consult

[1]Translator's note: "Un cours de troisième cycle" is equivalent to an advanced graduate course in an American university.

[2]Translator's note: These are now also called Gross inequalities.

[**Cor02, OR07**], and their bibliographies if they are interested in further study of the subjects treated here.

CHAPTER 1

# Self-Adjoint Operators

We denote by $H, (\cdot,\cdot)$ a separable complex Hilbert space,[1] by $\mathcal{D}$ a dense linear subspace of $H$, and by $A$ an operator from $\mathcal{D}$ to $H$. The space $\mathcal{D}$ is called the domain of the operator $A$ and is denoted $D(A)$. Unlike bounded operators,[2] in particular, operators on any finite dimensional Hilbert space, simple consideration of the symmetry of operators does not lead to a theorem of spectral decomposition. We will introduce directly the notion of self-adjointness by utilizing spectral conditions based on an exposé of P. Cartier at l'École Polytechnique.

## 1.1. Symmetric operators

***Definition* 1.1.1.** We say that the complex number $\lambda$ is in the resolvent set $\rho(A)$ of $A$ if $(\lambda \mathrm{Id} - A)$ is injective, its image $(\lambda \mathrm{Id} - A)\mathcal{D}$ is dense in $\mathcal{H}$, and if the inverse operator $(\lambda \,\mathrm{Id} - A)^{-1}$ is a bounded operator from $(\lambda\,\mathrm{Id} - A)\mathcal{D}$ to $H$. This operator is then uniquely extended to a bounded operator $R_\lambda$ on $H$ called the resolvent operator.

We often abbreviate $A - \lambda\,\mathrm{Id}$ by $A - \lambda$.

***Proposition* 1.1.2 (*Resolvent Equation*).** . *For all $\lambda$, $\mu \in \rho(A)$ we have:*

$$R_\lambda - R_\mu = (\lambda - \mu) R_\mu R_\lambda.$$

Note that the Resolvent Equation implies that $\{R_\lambda\}$ is a commutative family of operators.

***Definition* 1.1.3.** We say that $A$ is closed if $\mathcal{D}$ is complete for the norm

$$\|\psi\|_A = \left(\|\psi\|^2 + \|A\psi\|^2\right)^{1/2}.$$

Consider the graph of $A$:

$$\mathcal{G}_A = \{(\psi, A\psi) \in H \times H \ : \ \psi \in \mathcal{D}\}.$$

It is obvious that the projection of the graph of $A$, with the usual product norm $H \times H$, onto $\mathcal{D}$, with the norm $|\cdot|_A$, is an isometry. Thus it is clear that $A$ is closed if its graph $\mathcal{G}_A$ is closed in $H \times H$. For a closed operator $A$ one can express the resolvent set $\rho(A)$ in a simpler way.

---

[1]The scalar product is left linear and right anti-linear.

[2]Recall that an operator $B$ is bounded if there exists a constant $M$ such that $\|B\| \leq M\|x\|$ for all $x$ in $\mathcal{D}$.

***Proposition* 1.1.4.** *Let $A$ be a closed operator. In order for $\lambda$ to be in $\rho(A)$, it is necessary and sufficient that one of the two following conditions hold:*

(1) *The mapping $(\lambda - A)$ is a bijection of $\mathcal{D}$ onto $H$.*
(2) *There exists a bounded operator $R_\lambda$ of $H$ such that:*

$$\begin{cases} R_\lambda \circ (\lambda - A) = \mathrm{Id}_{\mathcal{D}} \\ (\lambda - A) \circ R_\lambda = \mathrm{Id}_H \,. \end{cases} \tag{1.1.1}$$

PROOF. (1) In order to show the necessity of the condition, we need to show that if $\lambda \in \rho(A)$ then $\mathrm{Image}(\lambda - A) = H$. Since this image is dense, there exists for any $x \in H$ a sequence $y_n$ of elements in $\mathcal{D}$ such that $x = \lim(\lambda y_n - Ay_n)$. By applying the bounded operator $R_\lambda$, we can conclude that $y_n = R_\lambda(\lambda - A)y_n$ converges. Since both $y_n$ and $Ay_n$ converge, and $\mathcal{G}_A$ is closed, the limit $y$ of $y_n$ is in $\mathcal{D}$ and $\lim(Ay_n) = Ay$ from which we conclude that $\lambda y - Ay = x$. Since $x$ is arbitrary, we see that $(\lambda - A)\mathcal{D} = H$.

Now suppose $\lambda - A$ is a bijection. It is a continuous mapping from the Hilbert space $(\mathcal{D}, \|\cdot\|_A)$ to the Hilbert space $H$. By Banach's open mapping theorem, the inverse mapping is continuous and obviously remains continuous if we equip $\mathcal{D}$ with the weaker norm $\|\cdot\|_H$.

(2) We see these conditions are equivalent to the initial definition if we take into account the fact that $\lambda - A$ is surjective if its image is dense. □

Self-adjoint operators are a special class of symmetric operators where by a symmetric operator $A$ with a dense domain $\mathcal{D}$ in $H$ we mean a linear operator $A : \mathcal{D} \longrightarrow H$ that satisfies:

$$\forall \varphi, \psi \in \mathcal{D} \quad \langle A\varphi, \psi\rangle = \langle \varphi, A\psi\rangle.$$

They are often defined on natural domains that are too small for the operator $A$ to be closed. A basic example is the Laplacian $\Delta$ defined, say, on $\mathcal{D} = C_c^\infty(\mathbb{R}^n)$, the space of infinitely differentiable functions on $\mathbb{R}^n$ with compact support. However these operators are easily seen to be closeable in the following sense:

***Proposition* 1.1.5.** *The closure of the graph of the symmetric operator $A$ with domain $\mathcal{D}$ as a subset of $H \times H$ is the graph of an operator $\overline{A}$ defined on a domain $\mathcal{D}' \supset \mathcal{D}$. Moreover the resolvent sets and the resolvent operators are the same for both operators. ($\overline{A}$ is called the closure of $A$.)*

PROOF. We first show that $\overline{\mathcal{G}_A}$ is a graph of a function from $H$ to $H$. We need to show that if $(\varphi, \psi) \in \overline{\mathcal{G}_A}$ and $(\varphi, \psi') \in \overline{\mathcal{G}_A}$ then $\psi = \psi'$. There exists a sequence $(\varphi_n, A\varphi_n)$ that converges to $(\varphi, \psi)$ and similarly a sequence $(\varphi_n', A\varphi_n')$ that converges to $(\varphi, \psi')$. Let $w$ be an arbitrary element of $\mathcal{D}$. Since

$$\langle w, \psi\rangle = \lim_{n\to\infty} \langle w, A\varphi_n\rangle = \lim_{n\to\infty} \langle Aw, \varphi_n\rangle = \langle Aw, \varphi\rangle,$$

and similarly since $\langle w, \psi'\rangle = \langle Aw, \varphi\rangle$ for all $w$ in the dense space $\mathcal{D}$, one necessarily has $\psi = \psi'$.

It is immediate by passing to the limit that the operator associated to the graph $\overline{\mathcal{G}_A}$ remains symmetric and that it is closed. We establish that $\rho(A) \subset \rho(\overline{A})$ as follows: for $\lambda \in \rho(A)$ the only thing that needs to be justified is that $\lambda - \overline{A}$ is injective. In fact, if $\varphi \in \ker(\lambda - \overline{A})$, there exists a sequence $\varphi_n$ that converges to $\varphi$ and such that $\psi_n = \lambda\varphi_n - A\varphi_n$ converges to 0, from which we have: $\varphi = \lim \varphi_n = \lim R_\lambda \psi_n = 0$.

We now consider the inclusion in the other direction. By construction, $\mathcal{D}$ is dense in $\mathcal{D}'$ for the norm $\|\cdot\|_{\overline{A}}$ and $A$ is continuous with respect to this norm. Using this, the property $\overline{(\lambda - A)\mathcal{D}} = H$ follows from the fact that $\overline{(\lambda - \overline{A})\mathcal{D}'} = H$. The other properties characterizing $\rho(A)$ follow immediately from the fact that it is contained in $\rho(\overline{A})$. □

The following lemma allows us to study *a priori* the resolvent set and the spectrum $\sigma(A) := \mathbb{C}\backslash\rho(A)$ of a symmetric operator $A$.

***Lemma* 1.1.6.** *Let $A$ be a closed operator and let $|\cdot|$ be the uniform norm on bounded operators on $H$.*

(1) *If $\lambda \in \rho(A)$, then the open disk $D(\lambda, \|R_\lambda\|^{-1})$ in $\mathbb{C}$ is contained in $\rho(A)$. In particular $\rho(A)$ is open.*
(2) *If $A$ is symmetric and $\lambda \in \rho(A)$, then the disk $D(\lambda, \Im(\lambda))$ is contained in $\rho(A)$.*

PROOF. (1) If $|\mu - \lambda| < \|R_\lambda\|^{-1}$, the series $S = \sum_{n=0}^{\infty} (\lambda - \mu)^n R_\lambda^n$ converges in the uniform norm on the algebra of bounded operators and $R_\lambda S$ is a bounded operator $R_\mu$ that satisfies the equations (1.1).

(2) If $\alpha$ and $\beta$ are two real numbers, we have:

$$\alpha + i\beta \in \rho(A) \text{ and } \beta \neq 0 \Rightarrow \|R_\lambda\| \leqslant \beta^{-1}. \tag{1.1.2}$$

This follows from the coercivity inequality

$$\|(A - \alpha - i\beta)x\|^2 \geqslant \beta^2\|x\|^2, \tag{1.1.3}$$

which in turn follows from the calculation, for $x \in \mathcal{D}$,

$$\begin{aligned}\|(A - \alpha - i\beta)x\|^2 &= \langle (A-\alpha)x, (A-\alpha)x\rangle + \langle \beta x, \beta x\rangle \\ &\quad + i\langle \beta x, (A-\alpha)x\rangle - i\langle (A-\alpha)x, \beta x\rangle \\ &= \|(A-\alpha)x\|^2 + \beta^2\|x\|^2.\end{aligned}$$

□

Let $\mathbb{C}^\pm := \{\lambda \in \mathbb{C} \ : \ \pm\Im(\lambda) \geq 0\}$. Then we have the following proposition, the proof of which is left as an exercise for the reader.

***Proposition* 1.1.7.** *There are only four mutually exclusive possibilities for the spectrum of a symmetric operator $A$: $\sigma(A)$ is equal to $\mathbb{C}^+$, $\mathbb{C}^-$, $\mathbb{C}$, or it is included in $\mathbb{R}$.*

***Definition* 1.1.8.** We say an operator $(A, \mathcal{D})$ is essentially self-adjoint if it is symmetric and if $\sigma(A) \subset \mathbb{R}$; if, in addition, it is closed we say it is self-adjoint. If $A$ is self-adjoint, a *core* for $A$ is any dense domain contained in the domain of $A$ on which the restriction of $A$ is essentially self-adjoint.

***Proposition* 1.1.9.** *Let $A$ be a symmetric (resp. symmetric and closed) operator on $\mathcal{D}$. Then (1) below is a sufficient condition that $A$ be essentially self-adjoint (resp. self-adjoint) and (2) and (3) are necessary and sufficient conditions that it be essentially self-adjoint (resp. self-adjoint).*

(1) *There exists a real number $\lambda$ in $\rho(A)$.*
(2) *$\pm i$ are in $\rho(A)$.*
(3) *The images $(i - A)\mathcal{D}$ and $(i + A)\mathcal{D}$ are dense.*

PROOF. The first two conditions allow us to eliminate the first three possibilities in Proposition 1.1.7. Finally, condition (3) is equivalent to (2). Indeed, the conditions of injectivity and of the continuity of the inverse, which are part of the definition of $R_i$ (or $R_{-i}$), are always satisfied in the symmetric case by applying the inequality (1.1.3). □

***Exercise* 1.1.10.** Verify that for $A$ symmetric and $\lambda$ real the necessary condition of injectivity for $\lambda \in \rho(A)$ is a consequence of the density condition.

***Exercise* 1.1.11.** Let $(W, \mathcal{F}, \mu)$ be a measure space and let $\chi$ be a real-valued measurable function on $W$.

Show that the possibly unbounded operator $M_\chi$ on $H = L^2(\mu)$ defined on $\mathcal{D} := \{\varphi \in H : \varphi\chi\} \in H$ by $M_\chi(\varphi) := \chi\varphi$ is self-adjoint and that the resolvent operator $R_\lambda$, when it exists, is multiplication by $(\lambda - \chi)^{-1}$.

Deduce from this that the spectrum of $M_\chi$ is "the essential image" of $\chi$, i.e., the support of the measure $\chi(\mu)$. By the support of a measure $\nu$ on $\mathbb{R}$, we mean the closed complement of the union of all open sets of $\nu$-measure zero.

***Exercise* 1.1.12.** Let $A$ and $B$ be two self-adjoint operators such that $A$ extends $B$, that is to say, $\mathcal{D}_A \supset \mathcal{D}_B$ and $B$ is the restriction of $A$ to $\mathcal{D}_B$. Show that $A = B$.

***Exercise* 1.1.13.** Consider $H = L^2([0,1])$ and let

$$\mathcal{D}_0 := \{f \in C^1([0,1]) \ : \ f(0) = f(1) = 0\}.$$

For $f \in \mathcal{D}_0$, we set $Af = if'$.

Verify that the operator $A$ is symmetric on $\mathcal{D}_0$. Verify that $(i-A)\mathcal{D}_0 \perp u$, where $u(x) = e^{-x}$, and deduce from this that $(A, \mathcal{D}_0)$ is not essentially self-adjoint. Show that any distribution $u$ that is the solution of the equation $u' = ku$ is equal to the function $c\,e^{kx}$.

Let $\alpha$ be a fixed complex number of modulus 1. Let $B$ be the operator defined by the same formula as $A$ but on the domain: $\mathcal{D} = \{f \in C^1([0,1]) \ : \ f(1) = \alpha f(0)\}$. Show that $B$ is essentially self-adjoint.

A particularly useful application of Proposition 1.1.9 is to symmetric operators that are bounded below. We say a symmetric operator $A$ is *positive* when:

$$\forall x \in \mathcal{D}(A) \qquad \langle Ax, x\rangle \geqslant 0,$$

and we say that $A$ is bounded below if there exists a constant $m$ such that $A - m\,\mathrm{Id}$ is positive. In this case we also say that $A$ is bounded below by $m$, i.e.,

$$\exists m\, \forall x \qquad \langle Ax, x\rangle \geqslant m\|x\|^2.$$

***Proposition* 1.1.14.** *Let $A$ be a symmetric operator that is bounded below by $m$ and $\lambda < m$. Then in order for $\lambda \in \rho(A)$, it is necessary and sufficient that $(\lambda - A)\mathcal{D}$ is dense. Thus if $(\lambda - A)\mathcal{D}$ is dense, $A$ is essentially self-adjoint on $\mathcal{D}$.*

PROOF. Since $\langle x, (A-\lambda)x\rangle \geqslant (m-\lambda)\|x\|^2$, we see right away that the condition of injectivity and the condition of continuity of $R_\lambda$ are satisfied. Thus by the definition of $\rho(A)$, $\lambda \in \rho(A)$ if and only if $(\lambda - A)\mathcal{D}$ is dense. By Proposition 1.1.9(1), $\lambda \in \rho(A)$ implies $A$ is essentially self-adjoint. □

***Proposition* 1.1.15.** *As an operator on the domain $\mathcal{D} = C_c^\infty(\mathbb{R}^n)$ in the Hilbert space $L^2(\mathbb{R}^n)$, the operator $\Delta$ is essentially self-adjoint.*

PROOF. It obviously is sufficient to show that $-\Delta$ is essentially self-adjoint on this domain. We first see that $-\Delta$ is symmetric and positive from Green's formula:

$$\langle -\Delta\varphi, \psi\rangle = \int_{\mathbb{R}^n} \nabla\varphi\nabla\overline{\psi}\, dx.$$

To show that it is essentially self-adjoint it is sufficient by Proposition 1.1.14 to show that $(\mathrm{Id} - \Delta)\mathcal{D}$ is dense. Indeed if this were not the case, there would exist a function $f \neq 0$ in $L^2$ such that $\langle f, \varphi - \Delta\varphi\rangle = 0$, for all $\varphi \in \mathcal{D}$. Utilizing the Fourier transform on $L^2(\mathbb{R}^n)$, which is an isometry, we would have $\langle \widehat{f}(p), (p^2+1)\widehat{\varphi}(p)\rangle = 0$. This in turn would imply that $\widehat{f} = 0$ since the subspace of functions

$$T = \{(p^2+1)\widehat{\varphi}(p) \,:\, \varphi \in \mathcal{D}\}$$

is dense in $L^2(\mathbb{R}^n)$. This contradicts the assumption that $f \neq 0$.

Here we recall a proof that $T$ is dense. Consider a function $\psi \in \mathcal{S}$ where $\mathcal{S}$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}$. One can find a sequence of functions $\varphi_n$ of $C_c^\infty$, such that for each order $\alpha$ the derivative $\varphi_n^\alpha$ tends to $\psi^\alpha$ in $L^2$. For example, $\varphi_n(x) = \varphi(x/n)\psi(x)$ where $\varphi \in C_c^\infty$ is equal to 1 in a neighborhood of 0. Thus $-\Delta\varphi_n + \varphi_n$ converges to $-\Delta\psi + \psi$ in $L^2$. Since the Fourier transform is an isometry on $L^2$, we see that $(p^2+1)\widehat{\psi}$ is in the closure of $T$. But any function in $\mathcal{S}$ can be written in the preceding form. Hence $\overline{T}$ contains $\overline{\mathcal{S}} = L^2$. □