A Collection of Matrices for Testing Computational Algorithms

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To Alston S. Householder

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PREFACE

This monograph is intended primarily as a reference book for numerical analysts and others who are interested in computational methods for solving problems in matrix algebra. It is well known that a good mathematical algorithm may or may not be a good computational algorithm. Consequently, what is needed is a collection of numerical examples with which to test each algorithm as soon as it is proposed. It is our hope that the matrices we have collected will help fulfill this need.

The test matrices in this collection were obtained for the most part by searching the current literature. However, four individuals who had begun collections of their own contributed greatly to this effort by providing a large number of test matrices at one time.

First, Joseph Elliott's Master's thesis [18] provided a large collection of tridiagonal matrices. Second, Mrs. Susan Voigt, of the Naval Ship Research and Development Center, contributed a varied collection of matrices. Third, Professor Robert E. Greenwood, of The University of Texas at Austin, provided a valuable list of references along with his collection of matrices and determinants. Finally, just as this work was nearing completion, the collection of Dr. Joan Westlake [60] was discovered. Her collection of 41 test matrices contained seven which we had overlooked; therefore, they were added.

Matrix 6.11 is a non-Hermitian matrix of order 20. It is a specific example of a class of matrices known as Dolph-Lewis matrices [14] which arose around 1957 in an investigation of perturbations of plane Poiseuille flow. Accurate eigenvalues, along with left and right eigenvectors and condition numbers, were provided by Dr. J. H. Wilkinson of the National Physical Laboratory.

Matrix 3.8 is the finite segment (of order n) of the (infinite) Hilbert Matrix. Matrix 3.26 is a generalization. The exact inverses of the finite Hilbert segments exhibited were provided by Dr. Max Engeli of FIDES Treuhand-Vereinigung, Zürich. Dr. Engeli's program for computing these inverses was written in SYMBAL, a language of his own creation.

The first author is grateful to Dr. Engeli and to Dr. Erwin Nievergelt for making the facilities of FIDES, including the CDC 6500 computer, available to him during his stay in Zürich.

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The book is dedicated to Dr. Alston S. Householder, who has inspired numerical analysts for the past two decades.

We are indebted to Mrs. Dorothy Baker for preparing the manuscript. Her superb job of typing this difficult material enabled the publishers to use photographic reproduction. This saved the authors an enormous amount of additional proofreading and avoided the introduction of countless additional errors.

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AUSTIN, TEXAS
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CHAPTER I

INTRODUCTION

In order to test the accuracy of computer programs for solving numerical problems, one needs numerical examples with known solutions. The aim of this monograph is to provide the reader with suitable examples for testing algorithms for finding the inverses, eigenvalues, and eigenvectors of matrices. A collection of methods for constructing test matrices and a large collection of numerical examples have been included. We have endeavored to allow the reader much freedom in his choice of a test matrix.

Chapter II of this monograph describes methods for generating matrices with known inverses and eigensystems whereas Chapter III contains test matrices with known inverses and solutions of systems of linear equations.

In the later chapters test matrices with known eigenvalues and eigenvectors are given. We have included, when possible, both right and left eigenvectors. The reader is reminded that if A is an Hermitian matrix, the left eigenvectors of A are the conjugate transpose of the right eigenvectors. For some of the examples, the tridiagonal forms are given which arise in the use of certain well-known algorithms for computing eigenvalues. The methods of Givens and Householder, for example, transform real symmetric matrices into the tridiagonal form

$$\begin{bmatrix} \alpha_1 & \beta_2 & & & \\ \beta_2 & \alpha_2 & \beta_3 & & \\ & \cdots & \cdots & \cdots & \\ & & \beta_{n-1} & \alpha_{n-1} & \beta_n \\ & & & \beta_n & \alpha_n \end{bmatrix} .$$

The method of Lanczos transforms nonsymmetric matrices into the tridiagonal form

$$\begin{bmatrix} \alpha_1 & \beta_2 & & & \\ 1 & \alpha_2 & \beta_3 & & \\ & \cdots & \cdots & \cdots & \\ & & 1 & \alpha_{n-1} & \beta_n \\ & & & 1 & \alpha_n \end{bmatrix} .$$

The examples exhibited in this monograph include both well-conditioned and ill-conditioned matrices. For each example we have computed several condition numbers, and for the ill-conditioned matrices the condition numbers are included.

Let $A = [a_{ij}]$ be an n x n nonsingular matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. For the problem of matrix inversion, at least three condition numbers are used. Von Neumann and Goldstine [59] suggest the condition number

$$P(A) = \frac{\max_i |\lambda_i|}{\min_i |\lambda_i|} .$$

Turing [57] proposes the two condition numbers

$$M(A) = n \max_{i,j} |a_{ij}| \max_{i,j} |\alpha_{ij}|$$

and

$$N(A) = \frac{1}{n} \|A\|_E \cdot \|A^{-1}\|_E ,$$

where

$$\|A\|_E = \left[\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right]^{\frac{1}{2}} ,$$

and where

$$A^{-1} = [\alpha_{ij}] .$$

It can be shown that P(A) and N(A) do not differ very much from M(A). In particular, we have [60, p. 90], [53]

$$\frac{1}{n^2} M(A) \leqq N(A) \leqq M(A)$$

and

$$P(A) \leqq nM(A) .$$

If A is symmetric, we also have

$$\frac{1}{n} M(A) \leqq P(A) .$$

If the matrix elements are chosen at random from a normal population, then an N-condition number of order $\sqrt{n}$ and an M-condition number of order $\sqrt{n} \log n$ can be expected.

Actually, M(A) and N(A) are not used as much as the more general condition number

$$K(A) = \|A\| \cdot \|A^{-1}\| ,$$

for various norms, not necessarily the same.

Now let $x^{(i)}$ and $y^{(i)}$ be, respectively, right and left eigenvectors of A corresponding to the eigenvalue λ_i, and suppose $x^{(i)}$ and $y^{(i)}$ are normalized so that

$$\sum_{j=1}^{n} |x_j^{(i)}|^2 = \sum_{j=1}^{n} |y_j^{(i)}|^2 = 1.$$

The condition of A with respect to the eigenvalue problem can be measured by the n condition numbers of A [62, pp. 88-89], $|s_i|^{-1}$, where

$$s_i = y^{(i)^T} x^{(i)}, \quad i = 1, 2, \ldots, n.$$

Here, $|s_i|^{-1}$ is the condition number for λ_i. Thus, some eigenvalues may be more ill-conditioned than others. Observe that if A is Hermitian, we have, for all i,

$$s_i = 1.$$

CHAPTER II

CONSTRUCTION OF TEST MATRICES

1. In this chapter we present a variety of methods by which the reader can construct matrices with known inverses, eigenvalues, and eigenvectors. We begin with the following well-known results which can be found in elementary texts on matrix algebra such as Hohn [28].

<u>Theorem 1</u>. The eigenvalues of A and A^T are the same.

<u>Theorem 2</u>. The eigenvalues of $\bar{A}$ and A^H are the conjugates of the eigenvalues of A.

<u>Theorem 3</u>. The eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A.

<u>Theorem 4</u>. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of an $n \times n$ matrix A and if $P(\alpha)$ is a polynomial, then the eigenvalues of $P(A)$ are $P(\lambda_1), P(\lambda_2), \ldots, P(\lambda_n)$. Further, if x is an eigenvector of A corresponding to the eigenvalue λ, then x is an eigenvector of $P(A)$ corresponding to $P(\lambda)$.

<u>Theorem 5</u>. The matrix

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

has the characteristic equation

$$|A-\lambda I| = \lambda^n + a_1\lambda^{n-1} + \dots + a_n = 0.$$

Theorem 6. If B is a non-singular matrix, then the eigenvalues of BAB^{-1} are the same as those of A. If x and y are, respectively, right and left eigenvectors of A corresponding to the eigenvalue λ, then Bx and yB^{-1} are respectively right and left eigenvectors of BAB^{-1} corresponding to the eigenvalue λ. If A is also non-singular, then $(BAB^{-1})^{-1} = BA^{-1}B^{-1}$.

2. One of the simplest methods of constructing test matrices is by forming composite matrices (some authors use compound matrices). In this regard we have the following.

Theorem 7 [2]. The eigenvalues of a block-diagonal matrix, diag $[A_1, A_2, \dots, A_k]$, are the eigenvalues of $A_1, A_2, \dots, A_k$.

Theorem 8 [28, pp. 81-82]. Suppose B is composed of submatrices of indicated orders,

$$B = \left[\begin{array}{c|c} \begin{matrix} A_{11} \\ (n \times n) \end{matrix} & \begin{matrix} A_{12} \\ (n \times m) \end{matrix} \\ \hline \begin{matrix} A_{21} \\ (m \times n) \end{matrix} & \begin{matrix} A_{22} \\ (m \times m) \end{matrix} \end{array}\right]$$

and suppose A_{11} and $P = A_{22} - A_{21}(A_{11}^{-1}A_{12})$ are non-singular. Then B is non-singular, and if we partition B^{-1} into submatrices

$$B^{-1} = \left[\begin{array}{c|c} B_{11} & B_{12} \\ (n \times n) & (n \times m) \\ \hline B_{21} & B_{22} \\ (m \times n) & (m \times m) \end{array}\right]$$

we have

$$B_{11} = A_{11}^{-1} + (A_{11}^{-1}A_{12})P^{-1}(A_{21}A_{11}^{-1})$$

$$B_{12} = -(A_{11}^{-1}A_{12})P^{-1}$$

$$B_{21} = -P^{-1}(A_{21}A_{11}^{-1})$$

$$B_{22} = P^{-1}.$$

Theorem 9 [36, p. 12]. If A and B are real $n \times n$ matrices and

$$S = \begin{bmatrix} A & B \\ B & A \end{bmatrix},$$

then the eigenvalues of S are the eigenvalues of $A + B$ together with the eigenvalues of $A - B$.

Another class of composite matrices suitable for test purposes can be obtained by the use of Kronecker products. Most of the following material comes from Bellman [2, Chapter 12] and Marcus [36]; the reader is referred to Friedman [25] for additional information.

Definition [2]. Let $A = [a_{ij}]$ be an $m \times m$ matrix and B an $n \times n$ matrix. The $mn \times mn$ matrix defined by

$$\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix}$$

is called the Kronecker product of A and B and is denoted by $A \otimes B$.

For matrices of this form we have the following very important results.

Theorem 10 [2]. If A is an m x m matrix with eigenvalues λ_i, i = 1, 2, ..., m, and B is an n x n matrix with eigenvalues μ_j, j = 1, 2, ..., n, then the eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$, i = 1, 2, ..., m and j = 1, 2, ..., n. The eigenvectors are mn x 1 column-vectors of the form

$$z_{ij} = \begin{bmatrix} x_1^{(i)} y^{(j)} \\ x_2^{(i)} y^{(j)} \\ \vdots \\ x_m^{(i)} y^{(j)} \end{bmatrix}$$

where $y^{(j)}$ is an eigenvector of B corresponding to the eigenvalue μ_j and $x_k^{(i)}$, k = 1, 2, ..., m, denote the components of the eigenvector $x^{(i)}$ of A corresponding to λ_i.

Theorem 11 [36, p. 5]. If A and B are non-singular, then $A \otimes B$ is non-singular and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

We can, of course, consider Kronecker powers of a particular matrix, i.e.,

$$A^{(2)} = A \otimes A$$
$$A^{(k+1)} = A \otimes A^{(k)}.$$

The eigenvalues of $A^{(k)}$ are all possible products consisting of k factors, each of which is an eigenvalue of A. We can also define matrices having eigenvalues of this form which are of much smaller dimension than the general Kronecker product. For example [2], suppose

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and suppose A has eigenvalues λ_1, λ_2. Starting with the equations

$$\lambda_1 x_1 = a_{11}x_1 + a_{12}x_2$$

$$\lambda_1 x_2 = a_{21}x_1 + a_{22}x_2 \;,$$

we form the products, for a fixed integer k,

$$(a_{11}x_1 + a_{12}x_2)^{k-i}(a_{21}x_1 + a_{22}x_2)^i, \quad i = 0, 1, \ldots, k.$$

Then, if we let $A_{(k)} = [b_{ij}]$, $i,j = 0, 1, \ldots, k$, denote the k+1 x k+1 matrix such that b_{ij} is the coefficient of the $x_1^{k-j}x_2^j$ term in the product $(a_{11}x_1 + a_{12}x_2)^{k-i}(a_{21}x_1 + a_{22}x_2)^i$, the eigenvalues of $A_{(k)}$ are $\lambda_1^{k-i}\lambda_2^i$, $i = 0, 1, \ldots, k$.

For example, if k = 2, we have the products

$$(a_{11}x_1 + a_{12}x_2)^2 = a_{11}^2x_1^2 + 2a_{11}a_{12}x_1x_2 + a_{12}^2x_2^2$$

$$(a_{11}x_1 + a_{12}x_2)(a_{21}x_1 + a_{22}x_2) = a_{11}a_{21}x_1^2 + (a_{11}a_{22} + a_{12}a_{21})x_1x_2 + a_{12}a_{22}x_2^2$$

$$(a_{21}x_1 + a_{22}x_2)^2 = a_{21}^2x_1^2 + 2a_{21}a_{22}x_1x_2 + a_{22}^2x_2^2 \;.$$

Thus the matrix

$$A_{(2)} = \begin{bmatrix} a_{11}^2 & 2a_{11}a_{12} & a_{12}^2 \\ a_{11}a_{21} & (a_{11}a_{22} + a_{12}a_{21}) & a_{12}a_{22} \\ a_{21}^2 & 2a_{21}a_{22} & a_{22}^2 \end{bmatrix}$$

has eigenvalues $\lambda_1^2, \lambda_1\lambda_2, \lambda_2^2$.

Next let us suppose that A is an m x m matrix with eigenvalues λ_i, $i = 1, 2, \ldots, m$, and B is an n x n matrix with eigenvalues μ_j, $j = 1, 2, \ldots, n$. It can be shown that the mn eigenvalues of $(I_m \otimes B) + (A \otimes I_n)$ are $\lambda_i + \mu_j$, for all i and j. For example, let A be the m x m matrix

$$A = \begin{bmatrix} a & b & & & & \\ b & a & b & & & \\ & b & a & b & & \\ & \cdots & \cdots & \cdots & \cdots & \\ & & & b & a & b \\ & & & & b & a \end{bmatrix}$$

and let B be the n x n matrix

$$B = \begin{bmatrix} d & c & & & & \\ c & d & c & & & \\ & c & d & c & & \\ & \cdots & \cdots & \cdots & \cdots & \\ & & & c & d & c \\ & & & & c & d \end{bmatrix}$$

where the eigenvalues of A are $a + 2b \cos \frac{k\pi}{m+1}$, $k = 1, 2, \ldots, m$, and those of B are $d + 2c \cos \frac{k\pi}{n+1}$, $k = 1, 2, \ldots, n$. Then the eigenvalues of

$$(I_m \otimes B) + (A \otimes I_n) = \begin{bmatrix} (aI_n + B) & bI_n & & & \\ bI_n & (aI_n + B) & bI_n & & \\ & bI_n & (aI_n + B) & bI_n & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & bI_n & (aI_n + B) \end{bmatrix}$$

are given by

$$\lambda_{ij} = a + d + 2b \cos \frac{i\pi}{m+1} + 2c \cos \frac{j\pi}{n+1},$$

$$i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n.$$

3. Determinants can also be used to define another type of matrix "power." For simplicity we shall consider a 3 x 3 matrix A and a set of 2 x 2 determinants formed from the eigenvectors of A. The procedure can be generalized to treat m x m determinants associated with the eigenvectors of n x n matrices [2].

Let

$$A = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

and let the eigenvalues of A be λ_1, λ_2, λ_3 with associated eigenvectors $x^{(1)}$, $x^{(2)}$, $x^{(3)}$. Now define, for arbitrary n-dimensional vectors r and s,

$$g_{ij}(r,s) = \det \begin{bmatrix} r_i & s_i \\ r_j & s_j \end{bmatrix}, \; i,j = 1, 2, \ldots, n.$$

Then, for our example, it can be shown that the matrix

$$G = \begin{bmatrix} g_{12}(a,b) & g_{23}(a,b) & g_{31}(a,b) \\ g_{12}(a,c) & g_{23}(a,c) & g_{31}(a,c) \\ g_{12}(b,c) & g_{23}(b,c) & g_{31}(b,c) \end{bmatrix}$$

has eigenvalues $\lambda_1\lambda_2$, $\lambda_1\lambda_3$, $\lambda_2\lambda_3$. Also, corresponding to the eigenvalue $\lambda_i\lambda_j$, $i \neq j$, there is an eigenvector

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

where

$$y_1 = g_{12}(x^{(i)},x^{(j)}),\ y_2 = g_{23}(x^{(i)},x^{(j)}),\ \text{and}\ y_3 = g_{31}(x^{(i)},x^{(j)}).$$

4. Brenner [7] has described another set of composite matrices which can be used to test inversion and eigenvalue routines. Let f_n denote the $n \times 1$ column-vector whose components are all 1's. For arbitrary integers n and k, let $J_{nk} = f_n f_k^T$, i.e., J_{nk} is the $n \times k$ matrix whose elements are all 1's. The matrix J_{nn} has the following properties: f_n is an eigenvector of J_{nn} corresponding to the eigenvalue $\lambda = n$; every vector orthogonal to f_n is an eigenvector of J_{nn} corresponding to the eigenvalue $\lambda = 0$. The eigenvalue $\lambda = 0$ has multiplicity $n - 1$, and its associated invariant space is spanned by the vectors $g_i = f_n - n e_n^i$, $i = 1, 2, \ldots, n-1$, where e_n^i is the $n \times 1$ column vector which has components δ_{ij}, $j = 1, 2, \ldots, n$. This leads us to the following result.