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Analysis I

Translated from the German by Gary Brookfield

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Preface

Logical thinking, the analysis of complex relationships, the recognition of underlying simple structures which are common to a multitude of problems — these are the skills which are needed to do mathematics, and their development is the main goal of mathematics education.

Of course, these skills cannot be learned 'in a vacuum'. Only a continuous struggle with concrete problems and a striving for deep understanding leads to success. A good measure of abstraction is needed to allow one to concentrate on the essential, without being distracted by appearances and irrelevancies.

The present book strives for clarity and transparency. Right from the beginning, it requires from the reader a willingness to deal with abstract concepts, as well as a considerable measure of self-initiative. For these efforts, the reader will be richly rewarded in his or her mathematical thinking abilities, and will possess the foundation needed for a deeper penetration into mathematics and its applications.

This book is the first volume of a three volume introduction to analysis. It developed from courses that the authors have taught over the last twenty six years at the Universities of Bochum, Kiel, Zurich, Basel and Kassel. Since we hope that this book will be used also for self-study and supplementary reading, we have included far more material than can be covered in a three semester sequence. This allows us to provide a wide overview of the subject and to present the many beautiful and important applications of the theory. We also demonstrate that mathematics possesses, not only elegance and inner beauty, but also provides efficient methods for the solution of concrete problems.

Analysis itself begins in Chapter II. In the first chapter we discuss quite thoroughly the construction of number systems and present the fundamentals of linear algebra. This chapter is particularly suited for self-study and provides practice in the logical deduction of theorems from simple hypotheses. Here, the key is to focus on the essential in a given situation, and to avoid making unjustified assumptions. An experienced instructor can easily choose suitable material from this chapter to make up a course, or can use this foundational material as its need arises in the study of later sections.

In this book, we have tried to lay a solid foundation for analysis on which the reader will be able to build in later forays into modern mathematics. Thus most concepts and definitions are presented, right from the beginning, in their general form — the form which is used in later investigations and in applications. This way the reader needs to learn each concept only once, and then with this basis, can progress directly to more advanced mathematics.

We refrain from providing here a detailed description of the contents of the three volumes and instead refer the reader to the introductions to each chapter, and to the detailed table of contents. We also wish to direct the reader's attention to the numerous exercises which appear at the end of each section. Doing these exercises is an absolute necessity for a thorough understanding of the material, and serves also as an effective check on the reader's mathematical progress.

In the writing of this first volume, we have profited from the constructive criticism of numerous colleagues and students. In particular, we would like to thank Peter Gabriel, Patrick Guidotti, Stephan Maier, Sandro Merino, Frank Weber, Bea Wollenmann, Bruno Scarpellini and, not the least, our students, who, by their positive reactions and later successes, encouraged our particular method of teaching analysis.

From Peter Gabriel we received support 'beyond the call of duty'. He wrote the appendix 'Introduction to Mathematical Logic' and unselfishly allowed it to be included in this book. For this we owe him special thanks.

As usual, a large part of the work necessary for the success of this book was done 'behind the scenes'. Of inestimable value are the contributions of our 'typesetting perfectionist' who spent innumerable hours in front of the computer screen and participated in many intense discussions about grammatical subtleties. The typesetting and layout of this book are entirely due to her, and she has earned our warmest thanks.

We also wish to thank Andreas who supplied us with latest versions of TEX[1] and stood ready to help with software and hardware problems.

Finally, we thank Thomas Hintermann for the encouragement to make our lectures accessible to a larger audience, and both Thomas Hintermann and Birkhäuser Verlag for a very pleasant collaboration.

Zurich and Kassel, June 1998 H. Amann and J. Escher

[1]The text was typeset using LATEX. For the graphs, CorelDRAW! and Maple were also used.

Preface to the second edition

In this new edition we have eliminated the errors and imprecise language that have been brought to our attention by attentive readers. Particularly valuable were the comments and suggestions of our colleagues H. Crauel and A. Ilchmann. All have our heartfelt thanks.

Zurich and Hannover, March 2002 H. Amann and J. Escher

Preface to the English translation

It is our pleasure to thank Gary Brookfield for his work in translating this book into English. As well as being able to preserve the 'spirit' of the German text, he also helped improve the mathematical content by pointing out inaccuracies in the original version and suggesting simpler and more lucid proofs in some places.

Zurich and Hannover, May 2004 H. Amann und J. Escher

Contents

Preface v

Chapter I Foundations

1 Fundamentals of Logic 3

2 Sets 8

Elementary Facts 8
The Power Set 9
Complement, Intersection and Union 9
Products 10
Families of Sets 12

3 Functions 15

Simple Examples 16
Composition of Functions 17
Commutative Diagrams 17
Injections, Surjections and Bijections 18
Inverse Functions 19
Set Valued Functions 20

4 Relations and Operations 22

Equivalence Relations 22
Order Relations 23
Operations 26

5 The Natural Numbers 29

The Peano Axioms 29
The Arithmetic of Natural Numbers 31
The Division Algorithm 34
The Induction Principle 35
Recursive Definitions 39

6 Countability . . . 46
Permutations . . . 47
Equinumerous Sets . . . 47
Countable Sets . . . 48
Infinite Products . . . 49

7 Groups and Homomorphisms . . . 52
Groups . . . 52
Subgroups . . . 54
Cosets . . . 55
Homomorphisms . . . 56
Isomorphisms . . . 58

8 Rings, Fields and Polynomials . . . 62
Rings . . . 62
The Binomial Theorem . . . 65
The Multinomial Theorem . . . 65
Fields . . . 67
Ordered Fields . . . 69
Formal Power Series . . . 71
Polynomials . . . 73
Polynomial Functions . . . 75
Division of Polynomials . . . 76
Linear Factors . . . 77
Polynomials in Several Indeterminates . . . 78

9 The Rational Numbers . . . 84
The Integers . . . 84
The Rational Numbers . . . 85
Rational Zeros of Polynomials . . . 88
Square Roots . . . 88

10 The Real Numbers . . . 91
Order Completeness . . . 91
Dedekind's Construction of the Real Numbers . . . 92
The Natural Order on $\mathbb{R}$ . . . 94
The Extended Number Line . . . 94
A Characterization of Supremum and Infimum . . . 95
The Archimedean Property . . . 96
The Density of the Rational Numbers in $\mathbb{R}$ . . . 96
n^{th} Roots . . . 97
The Density of the Irrational Numbers in $\mathbb{R}$ . . . 99
Intervals . . . 100

11 The Complex Numbers . . . 103

Constructing the Complex Numbers . . . 103
Elementary Properties . . . 104
Computation with Complex Numbers . . . 106
Balls in $\mathbb{K}$ . . . 108

12 Vector Spaces, Affine Spaces and Algebras . . . 111

Vector Spaces . . . 111
Linear Functions . . . 112
Vector Space Bases . . . 115
Affine Spaces . . . 117
Affine Functions . . . 119
Polynomial Interpolation . . . 120
Algebras . . . 122
Difference Operators and Summation Formulas . . . 123
Newton Interpolation Polynomials . . . 124

Chapter II Convergence

1 Convergence of Sequences . . . 131

Sequences . . . 131
Metric Spaces . . . 132
Cluster Points . . . 134
Convergence . . . 135
Bounded Sets . . . 137
Uniqueness of the Limit . . . 137
Subsequences . . . 138

2 Real and Complex Sequences . . . 141

Null Sequences . . . 141
Elementary Rules . . . 141
The Comparison Test . . . 143
Complex Sequences . . . 144

3 Normed Vector Spaces . . . 148

Norms . . . 148
Balls . . . 149
Bounded Sets . . . 150
Examples . . . 150
The Space of Bounded Functions . . . 151
Inner Product Spaces . . . 153
The Cauchy-Schwarz Inequality . . . 154
Euclidean Spaces . . . 156
Equivalent Norms . . . 157
Convergence in Product Spaces . . . 159

4 **Monotone Sequences** . . . 163
Bounded Monotone Sequences . . . 163
Some Important Limits . . . 164

5 **Infinite Limits** . . . 169
Convergence to $\pm\infty$ . . . 169
The Limit Superior and Limit Inferior . . . 170
The Bolzano-Weierstrass Theorem . . . 172

6 **Completeness** . . . 175
Cauchy Sequences . . . 175
Banach Spaces . . . 176
Cantor's Construction of the Real Numbers . . . 177

7 **Series** . . . 183
Convergence of Series . . . 183
Harmonic and Geometric Series . . . 184
Calculating with Series . . . 185
Convergence Tests . . . 185
Alternating Series . . . 186
Decimal, Binary and Other Representations of Real Numbers . . . 187
The Uncountability of $\mathbb{R}$ . . . 192

8 **Absolute Convergence** . . . 195
Majorant, Root and Ratio Tests . . . 196
The Exponential Function . . . 199
Rearrangements of Series . . . 199
Double Series . . . 201
Cauchy Products . . . 204

9 **Power Series** . . . 210
The Radius of Convergence . . . 211
Addition and Multiplication of Power Series . . . 213
The Uniqueness of Power Series Representations . . . 214

Chapter III Continuous Functions

1 **Continuity** . . . 219
Elementary Properties and Examples . . . 219
Sequential Continuity . . . 224
Addition and Multiplication of Continuous Functions . . . 224
One-Sided Continuity . . . 228

2 The Fundamentals of Topology 232
Open Sets 232
Closed Sets 233
The Closure of a Set 235
The Interior of a Set 236
The Boundary of a Set 237
The Hausdorff Condition 237
Examples 238
A Characterization of Continuous Functions 239
Continuous Extensions 241
Relative Topology 244
General Topological Spaces 245

3 Compactness 250
Covers 250
A Characterization of Compact Sets 251
Sequential Compactness 252
Continuous Functions on Compact Spaces 252
The Extreme Value Theorem 253
Total Boundedness 256
Uniform Continuity 258
Compactness in General Topological Spaces 259

4 Connectivity 263
Definition and Basic Properties 263
Connectivity in $\mathbb{R}$ 264
The Generalized Intermediate Value Theorem 265
Path Connectivity 265
Connectivity in General Topological Spaces 268

5 Functions on $\mathbb{R}$ 271
Bolzano's Intermediate Value Theorem 271
Monotone Functions 272
Continuous Monotone Functions 274

6 The Exponential and Related Functions 277
Euler's Formula 277
The Real Exponential Function 280
The Logarithm and Power Functions 281
The Exponential Function on $i\mathbb{R}$ 283
The Definition of π and its Consequences 285
The Tangent and Cotangent Functions 289
The Complex Exponential Function 290
Polar Coordinates 291

Complex Logarithms 293
Complex Powers 294
A Further Representation of the Exponential Function 295

Chapter IV Differentiation in One Variable

1 Differentiability 301
The Derivative 301
Linear Approximation 302
Rules for Differentiation 304
The Chain Rule 305
Inverse Functions 306
Differentiable Functions 307
Higher Derivatives 307
One-Sided Differentiability 313

2 The Mean Value Theorem and its Applications 317
Extrema 317
The Mean Value Theorem 318
Monotonicity and Differentiability 319
Convexity and Differentiability 322
The Inequalities of Young, Hölder and Minkowski 325
The Mean Value Theorem for Vector Valued Functions 328
The Second Mean Value Theorem 329
L'Hospital's Rule 330

3 Taylor's Theorem 335
The Landau Symbol 335
Taylor's Formula 336
Taylor Polynomials and Taylor Series 338
The Remainder Function in the Real Case 340
Polynomial Interpolation 344
Higher Order Difference Quotients 345

4 Iterative Procedures 350
Fixed Points and Contractions 350
The Banach Fixed Point Theorem 351
Newton's Method 355

Chapter V Sequences of Functions

1 Uniform Convergence . . . 363

Pointwise Convergence . . . 363
Uniform Convergence . . . 364
Series of Functions . . . 366
The Weierstrass Majorant Criterion . . . 367

2 Continuity and Differentiability for Sequences of Functions . . . 370

Continuity . . . 370
Locally Uniform Convergence . . . 370
The Banach Space of Bounded Continuous Functions . . . 372
Differentiability . . . 373

3 Analytic Functions . . . 377

Differentiability of Power Series . . . 377
Analyticity . . . 378
Antiderivatives of Analytic Functions . . . 380
The Power Series Expansion of the Logarithm . . . 381
The Binomial Series . . . 382
The Identity Theorem for Analytic Functions . . . 386

4 Polynomial Approximation . . . 390

Banach Algebras . . . 390
Density and Separability . . . 391
The Stone-Weierstrass Theorem . . . 393
Trigonometric Polynomials . . . 396
Periodic Functions . . . 398
The Trigonometric Approximation Theorem . . . 401

Appendix Introduction to Mathematical Logic . . . 405

Bibliography . . . 411

Index . . . 413

Chapter I

Foundations

Most of this first chapter is about numbers — natural numbers, integers, real numbers and complex numbers. Without a clear understanding of these numbers, a deep investigation of mathematics is not possible. This makes a thorough discussion of number systems absolutely necessary.

To that end we have chosen to present a constructive formulation of these number systems. Starting with the Peano axioms for the natural numbers, we construct successively the integers, the rational numbers, the real numbers and finally, the complex numbers. At each step, we are guided by a desire to solve certain 'naturally' occurring equations. These constructions are relatively long and require considerable stamina from the reader, but those readers who persevere will be rewarded with considerable practice in mathematical thinking.

Even before we can talk about the natural numbers, the simplest of all number systems, we must consider some of the fundamentals of set theory. Here the main goal is to develop a precise mathematical language. The axiomatic foundations of logic and set theory are beyond the scope of this book.

The reader may well be familiar with some of the material in Sections 1–4. Even so, we have deliberately avoided appealing to the reader's intuitions and previous experience, and have instead chosen a relatively abstract framework for our presentation. In particular, we have been strict about avoiding any concepts that are not already precisely defined, and using claims that are not previously proved. It is important that, right from the beginning, students learn to work with definitions and derive theorems from them without introducing spurious additional assumptions.

The transition from the simplest number system, the natural numbers, to the most complicated number system, the complex numbers, is paralleled by a corresponding increasing complexity in the algebra needed. Therefore, in Sections 7–8 we discuss fairly thoroughly the most important concepts of algebra. Here again we have chosen an abstract approach with the goal that beginning students become familiar with certain mathematical structures which appear in later chapters of this book and, in fact, throughout mathematics.

A deeper understanding of these concepts is the goal of (linear) algebra and, in the corresponding literature, the reader will find many other applications. The goal of algebra is to derive rules which hold in systems satisfying certain small sets of axioms. The discovery that these axioms hold in complex problems of analysis will enable us to recognize underlying unity in diverse situations and to maintain an overview of an otherwise unwieldy area of mathematics. In addition, the reader should see early on that mathematics is a whole — it is not made up of disjoint research areas, isolated from each other.

Since the beginner usually studies linear algebra in parallel with an introduction to analysis, we have restricted our discussion of algebra to the essentials. In the choice of the concepts to present we have been guided by the needs of later chapters. This is particularly true about the material in Section 12, namely vector spaces and algebras. These we will meet frequently, for example, in the form of function algebras, as we penetrate further into analysis.

The somewhat 'dry' material of this first chapter is made more palatable by the inclusion of many applications. Since, as already mentioned, we want to train the reader to use only what has previously been proved, we are limited at first to very simple 'internal' examples. In later sections this becomes less of a restriction, as, for example, the discussion of the interpolation problems in Section 12 shows.

We remind the reader that this book is intended to be used either as a textbook for a course on analysis, or for self study. For this reason, in this first chapter, we are more thorough and cover more material than is possible in lectures. We encourage the reader to work through these 'foundations' with diligence. In the first reading, the proofs of Theorems 5.3, 9.1, 9.2 and 10.4 can be skipped. At a later time, when the reader is more comfortable with proofs, these gaps should filled.

1 Fundamentals of Logic

To make complicated mathematical relationships clear it is convenient to use the notation of symbolic logic. Symbolic logic is about **statements** which one can meaningfully claim to be true or false. That is, each statement has the *truth value* 'true' (T) or 'false' (F). There are no other possibilities, and no statement can be both true and false.

Examples of statements are 'It is raining', 'There are clouds in the sky', and 'All readers of this book find it to be excellent'. On the other hand, 'This sentence is false' is not a statement. Indeed, if the sentence were true, then it says that it is false, and if it is false, it follows that the sentence is true.

Any statement A has a **negation** $\neg A$ ('not A') defined by $\neg A$ is true if A is false, and $\neg A$ is false if A is true. We can represent this relationship in a *truth table*:

A	T	F
$\neg A$	F	T

Of course, in normal language 'not A' can be expressed in many ways. For example, if A is the statement 'There are clouds in the sky', then $\neg A$ could be expressed as 'There are no clouds in the sky'. The negation of the statement 'All readers of this book find it to be excellent' is 'There is at least one reader of this book who finds that it is not excellent' (but not 'No readers of this book find it to be excellent').

Two statements, A and B, can be combined using **conjunction** $\wedge$ and **disjunction** $\vee$ to make new statements. The statement $A \wedge B$ ('A and B') is true if both A and B are true, and is false in all other cases. The statement $A \vee B$ ('A or B') is false when both A and B are false, and is true in all other cases. The following truth table makes the definitions clear:

A	B	$A \wedge B$	$A \vee B$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Note that the 'or' of disjunction has the meaning 'and/or', that is, 'A or B' is true if A is true, if B is true, or if both A and B are true.

If $E(x)$ is an expression which becomes a statement when x is replaced by an object (member, thing) of a specified class (collection, universe) of objects, then E is a **property**. The sentence 'x has property E' means '$E(x)$ is true'. If x belongs to a class X, that is, x is an **element** of X, then we write $x \in X$, otherwise[1] $x \notin X$.

[1]It is usual when abbreviating statements with symbols (such as $\in$, $=$, etc.) to denote their negations using the corresponding slashed symbol ($\notin$, $\neq$, etc.).

Then

$$\{\, x \in X \ ;\ E(x) \,\}$$

is the class of all elements x of the collection X which have property E. If X is the class of all readers of this book and $E(x)$ is the statement 'x wears glasses', then $\{\, x \in X \ ;\ E(x) \,\}$ is the class of all readers of this book who wear glasses.

We write $\exists$ for the **quantifier** 'there exists'. The expression

$$\exists\, x \in X : E(x)$$

has the meaning 'There is (at least) one object x in (the class) X which has property E'. We write $\exists!\, x \in X : E(x)$ when exactly one such object exists.

We use the symbol $\forall$ for the quantifier 'for all'. Once again, in normal language statements containing $\forall$ can be expressed in various ways. For example,

$$\forall\, x \in X : E(x) \tag{1.1}$$

means that 'For each (object) x in (the class) X, the statement $E(x)$ is true', or 'Every x in X has the property E'. The statement (1.1) can also be written as

$$E(x)\ , \qquad \forall\, x \in X\ , \tag{1.2}$$

that is, 'Property E is true for all x in X'. In a statement such as (1.2) we usually leave out the quantifier $\forall$ and write simply

$$E(x)\ , \qquad x \in X\ . \tag{1.3}$$

Finally, we use the symbol $:=$ to mean 'is defined by'. Thus

$$a := b\ ,$$

means that the object (or symbol) a is defined by the object (or expression) b. One says also 'a is a new name for b' or 'a stands for b '. Of course $a = b$ means that objects a and b are equal, that is, a and b are simply different representations of the same object (statement, etc.).

1.1 Examples Let A and B be statements, X and Y classes of objects, and E a property. Then, using truth tables or other methods, one can easily verify the following statements:

(a) $\neg\neg A := \neg(\neg A) = A$.

(b) $\neg(A \wedge B) = (\neg A) \vee (\neg B)$.

(c) $\neg(A \vee B) = (\neg A) \wedge (\neg B)$.

(d) $\neg\big(\forall\, x \in X : E(x)\big) = \big(\exists\, x \in X : \neg E(x)\big)$. Example: The negation of the statement 'Every reader of this book wears glasses' is 'At least one reader of this book does not wear glasses'.

(e) $\neg\big(\exists\, x \in X : E(x)\big) = \big(\forall\, x \in X : \neg E(x)\big)$. Example: The negation of the statement 'There is a bald man in London' is 'No man in London is bald'.

(f) $\neg\big(\forall\, x \in X : \big(\exists\, y \in Y : E(x,y)\big)\big) = \big(\exists\, x \in X : \big(\forall\, y \in Y : \neg E(x,y)\big)\big)$.
Example: The negation of the statement 'Each reader of this book finds at least one sentence in Chapter I which is trivial' is 'At least one reader of this book finds every sentence of Chapter I nontrivial'.

(g) $\neg\big(\exists\, x \in X : \big(\forall\, y \in Y : E(x,y)\big)\big) = \big(\forall\, x \in X : \big(\exists\, y \in Y : \neg E(x,y)\big)\big)$.
Example: The negation of the statement 'There is a Londoner who is a friend of every New Yorker' is 'For each Londoner there is at least one New Yorker who is not his/her friend'. ■[2]

1.2 Remarks **(a)** For clarity, in the above examples, we have been careful to include all possible parentheses. This practice is to be recommended for complicated statements. On the other hand, statements are often easier to understand without parentheses and even without the membership symbol $\in$, so long as no ambiguity arises. In all cases, it is the order of the quantifiers that is significant. Thus '$\forall\, x\ \exists\, y : E(x,y)$' and '$\exists\, y\ \forall\, x : E(x,y)$' are different statements: In the first case, for all x there is some y such that $E(x,y)$ is true. Thus y depends on x, that is, for each x one has to find a (possibly) different y such that $E(x,y)$ is true. In the second case it suffices to find a fixed y such that the statement $E(x,y)$ is true for all x. For example, if $E(x,y)$ is the statement 'Reader x of this book finds the mathematical concept y to be trivial', then the first statement is 'Each reader of this book finds at least one mathematical concept to be trivial'. The second statement is 'There is a mathematical concept which every reader of this book finds to be trivial'.

(b) Using the quantifiers $\exists$ and $\forall$, negation becomes a purely 'mechanical' process in which the symbols $\exists$ and $\forall$ (as well as $\wedge$ and $\vee$) are interchanged (without changing the order) and statements which appear are negated (see Examples 1.1). For example, the negation of the statement '$\forall\, x\ \exists\, y\ \forall\, z : E(x,y,z)$' is '$\exists\, x\ \forall\, y\ \exists\, z : \neg E(x,y,z)$'. ■

Let A and B be statements. Then one can define a new statement, the **implication** $A \Rightarrow B$, ('A implies B') as follows:

$$(A \Rightarrow B) := (\neg A) \vee B\ . \tag{1.4}$$

Thus $A \Rightarrow B$ is false if A is true and B is false, and is true in all other cases (see Examples 1.1(a), (c)). In other words, $A \Rightarrow B$ is true when A and B are both true, or when A is false (independent of whether B is true or false). This means that a true statement cannot imply a false statement, and also that a false

[2]We use a black square to indicate the end of a list of examples or remarks, or the end of a proof.

statement implies any statement — true or false. It is common to express $A \Rightarrow B$ as 'To prove B it **suffices** to prove A', or 'B is **necessary** for A to be true', in other words, A is a **sufficient condition** for B, and B is a **necessary condition** for A.

The **equivalence** $A \Leftrightarrow B$ ('A and B are equivalent') of the statements A and B is defined by

$$(A \Leftrightarrow B) := (A \Rightarrow B) \wedge (B \Rightarrow A) .$$

Thus the statements A and B are equivalent when both $A \Rightarrow B$ and its **converse** $B \Rightarrow A$ are true, or when A is a **necessary and sufficient** condition for B (or vice versa). Another common way of expressing this equivalence is to say 'A is true **if and only if** B is true'.

A fundamental observation is that

$$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A) . \tag{1.5}$$

This follows directly from (1.4) and Example 1.1(a). The statement $\neg B \Rightarrow \neg A$ is called the **contrapositive** of the statement $A \Rightarrow B$.

If, for example, A is the statement 'There are clouds in the sky' and B is the statement 'It is raining', then $B \Rightarrow A$ is the statement 'If it is raining, then there are clouds in the sky'. Its contrapositive is, 'If there are no clouds in the sky, then it is not raining'.

If $B \Rightarrow A$ is true it does not, in general, follow that $\neg B \Rightarrow \neg A$ is true! Even when 'it is not raining', it is possible that 'there are clouds in the sky'.

To define a statement A so that it is true whenever the statement B is true, we write

$$A :\Leftrightarrow B$$

and say 'A is true, by definition, if B is true'.

In mathematics a true statement is often called a proposition, theorem, lemma or corollary.[3] Especially common are propositions of the form $A \Rightarrow B$. Since this statement is automatically true if A is false, the only interesting case is when A is true. Thus to prove that $A \Rightarrow B$ is true, one supposes that A is true and then shows that B is true.

The proof can proceed directly or 'by contradiction'. In the first case, one can use the fact (which the reader can easily check) that

$$(A \Rightarrow C) \wedge (C \Rightarrow B) \Rightarrow (A \Rightarrow B) . \tag{1.6}$$

If the statements $A \Rightarrow C$ and $C \Rightarrow B$ are already known to be true, then, by (1.6), $A \Rightarrow B$ is also true. If $A \Rightarrow C$ and $C \Rightarrow B$ are not known to be true and the

[3] All theorems, lemmas and corollaries are propositions. A theorem is a particularly important proposition. A lemma is a proposition which precedes a theorem and is needed for its proof. A corollary is a proposition which follows directly from a theorem.

implications $A \Rightarrow C$ and $C \Rightarrow B$ can be similarly decomposed, this procedure can be used to show $A \Rightarrow C$ and $C \Rightarrow B$ are true.

For a proof by contradiction one supposes that B is false, that is, $\neg B$ is true. Then one proves, using also the assumption that A is true, a statement C which is already known to be false. It follows from this 'contradiction' that $\neg B$ cannot be true, and hence that B is true.

Instead of $A \Rightarrow B$, it is often easier to prove its contrapositive $\neg B \Rightarrow \neg A$. According to (1.5) these statements are equivalent, that is, one is true if and only if the other is true.

At this point, we prefer not to provide examples of the above concepts since they would be necessarily rather contrived. Instead the reader is encouraged to identify these structures in the proofs in following section (see, in particular, the proof of Proposition 2.6).

The preceding discussion is incomplete in that we have neither defined the word 'statement' nor explained how to tell whether a statement is true or false. A further difficultly lies in our use of the English language, which, like most languages, contains many sentences whose meaning is ambiguous. Such sentences cannot be considered to be statements in the sense of this section.

For a more solid understanding of the rules of deduction, one needs mathematical logic. This provides a formal language in which the only statements appearing are those which can be derived from a given system of 'axioms' by means of well defined constructions. These axioms are 'unprovable' statements which are recognized as fundamental universal truths.

We do not wish to go further here into such formal systems. Instead, interested readers are directed to the appendix, 'Introduction to Mathematical Logic', which contains a more precise presentation of these ideas.

Exercises

1 "The Simpsons are coming to visit this evening," announced Maud Flanders. "The whole family — Homer, Marge and their three kids, Bart, Lisa and Maggie?" asked Ned Flanders dismayed. Maud, who never misses a chance to stimulate her husband's logical thinking, replied, "I'll explain it this way: If Homer comes then he will bring Marge too. At least one of the two children, Maggie and Lisa, are coming. Either Marge or Bart is coming, but not both. Either both Bart and Lisa are coming or neither is coming. And if Maggie comes, then Lisa and Homer are coming too. So now you know who is visiting this evening."

Who is coming to visit?

2 In the library of Count Dracula no two books contain exactly the same number of words. The number of books is greater than the total number of words in all the books. These statements suffice to determine the content of at least one book in Count Dracula's library. What is in this book?

2 Sets

Even though the reader is probably familiar with basic set theory, we review in this section some of the relevant concepts and notation.

Elementary Facts

If X and Y are sets, then $X \subseteq Y$ ('X is a **subset** of Y' or 'X is contained in Y') means that each element of X is also an element of Y, that is, $\forall\, x \in X : x \in Y$. Sometimes it is convenient to write $Y \supseteq X$ ('Y contains X') instead of $X \subseteq Y$. Equality of sets is defined by

$$X = Y :\Longleftrightarrow (X \subseteq Y) \wedge (Y \subseteq X) \ .$$

The statements

$$X \subseteq X \qquad \text{(reflexivity)}$$
$$(X \subseteq Y) \wedge (Y \subseteq Z) \Rightarrow (X \subseteq Z) \qquad \text{(transitivity)}$$

are obvious. If $X \subseteq Y$ and $X \neq Y$, then X is called a **proper subset** of Y. We denote this relationship by $X \subset Y$ or $Y \supset X$ and say 'X is properly contained in Y'.

If X is a set and E is a property then $\{\, x \in X \ ;\ E(x) \,\}$ is the subset of X consisting of all elements x of X such that $E(x)$ is true. The set

$$\emptyset_X := \{\, x \in X \ ;\ x \neq x \,\}$$

is the **empty subset** of X.

2.1 Remarks **(a)** Let E be a property. Then

$$x \in \emptyset_X \Rightarrow E(x)$$

is true for each $x \in X$ ('The empty set possesses every property').

Proof From (1.4) we have

$$\bigl(x \in \emptyset_X \Rightarrow E(x)\bigr) = \neg(x \in \emptyset_X) \vee E(x) \ .$$

The negation $\neg(x \in \emptyset_X)$ is true for each $x \in X$. ■

(b) If X and Y are sets, then $\emptyset_X = \emptyset_Y$, that is, there is exactly one **empty set**. This set is denoted $\emptyset$ and is a subset of any set.

Proof From (a) we get $x \in \emptyset_X \Rightarrow x \in \emptyset_Y$, hence $\emptyset_X \subseteq \emptyset_Y$. By symmetry, $\emptyset_Y \subseteq \emptyset_X$, and so $\emptyset_X = \emptyset_Y$. ■

The set containing the single element x is denoted $\{x\}$. Similarly, the set consisting of the elements a, b, ..., $*$, $\odot$ is written $\{a, b, \ldots, *, \odot\}$.