MODULAR MATHEMATICS

Analysis

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Series Preface

This series is designed particularly, but not exclusively, for students reading degree programmes based on semester-long modules. Each text will cover the essential core of an area of mathematics and lay the foundation for further study in that area. Some texts may include more material than can be comfortably covered in a single module, the intention there being that the topics to be studied can be selected to meet the needs of the student. Historical contexts, real life situations, and linkages with other areas of mathematics and more advanced topics are included. Traditional worked examples and exercises are augmented by more open-ended exercises and tutorial problems suitable for group work or self-study. Where appropriate, the use of computer packages is encouraged. The first level texts assume only the A-level core curriculum.

Professor, Chris D. Collinson, Dr. Johnston Anderson and Mr., Peter Holmes

Preface

In the past decade, ‘friendly’ introductions to Mathematical Analysis have appeared with increasing regularity, arguably in response to the perception that the rigour and abstraction demanded by the subject no longer feature strongly in school syllabi. Most mathematicians have little doubt that, despite its beauty and elegance (or even because of these?) Analysis is among the more challenging topics faced by new undergraduates. It is certainly all too easy to take it too far too quickly!

I hope that, despite a few indulgences in Tutorial Problems and occasionally in the Exercises, this will not be the verdict on the current text. I have tried to build on the basic concepts provided by a careful discussion of *convergence*, and have stuck resolutely to sequences where possible. The text should be seen as following the foundations laid by Keith Hirst’s *Numbers, Sequences and Series*, but is also designed to be fairly self-contained.

Elementary results and examples regarding convergence are reviewed in Chapter 2, while the consequences of the completeness axiom for the real numbers are developed in more detail in Chapter 3. Similarly, power series are used to provide a bridge between these techniques and the elementary transcendental functions in Chapter 4.

The main part of the book concerns the central topics of *continuity, differentiation* and *integration* of real functions. Here, too, sequential techniques are exploited where possible, and attention is given to the systematic approximation of complex structures by simpler ones. The ‘dynamic’ features of a sequential approach to topological ideas, which are so evident in the historical development of Analysis, make the subject more intuitive and accessible to the new student in my view.

Throughout, I have tried to highlight the historical context in which the subject has developed (not all that long ago!) and have paid attention to showing how increasing precision allows us to refine our geometric intuition. The intention is to stimulate the reader to reflect on the underlying concepts and ideas, rather than to ‘learn’ definitions which appear to come out of the blue. This means that the ‘shortest route’ to a major theorem is not always the most illuminating one, but the detours seem worthwhile if they increase understanding.

Much of the text is based on a one-semester module given to first-year students at Hull, who would typically have had some exposure to sequences and series already. The rewards in terms of student enlightenment are seldom immediate, but they are always enjoyable when they appear!

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I am indebted to Peter Beckett for supplying the Pascal programs, the essential parts of which appear in the Appendix and which I have inexpertly tried out on my classes. Many thanks to Neil Gordon for producing the Appendix in its current form and for his expert help in creating the figures and diagrams which appear in the text. Thanks also to Marek Capinski for his patient explanation of the subtleties of LaTex, which I hope to master one day, and to David Ross of Arnold for his help and encouragement. Above all I want to thank my wife, Heather, for her unfailing support and patience. This book is dedicated to her.

Ekkehard Kopp, *Hull, February 1996*

1

Introduction: Why We Study Analysis

Why study Analysis? Or better still: why prove anything? The question is a serious one, and deserves a careful answer. When computer graphics can illustrate the behaviour of even very complicated functions much more precisely than we can draw them ourselves, when chaotic motion can be studied in great detail, and seems to model many physical phenomena quite adequately, why do we still insist on producing ‘solid foundations’ for the Calculus? Its methods may have been controversial in the seventeenth century, when they were first introduced by Newton and Leibniz, but now it is surely an accepted part of advanced school mathematics and nothing further needs to be said? So, when the stated purpose of most introductory Analysis modules is to ‘justify’ the operations of the Calculus, students frequently wonder what all the fuss is about – especially when they find that familiar material is presented in what seem to be very abstract definitions, theorems and proofs, and familiar ideas are described in a highly unfamiliar way, all apparently designed to confuse and undermine what they already know!

1.1 What the computer cannot see …

Before you shut the book and file its contents under ‘useless pedantry’, let’s reflect a little on what we *know* about the Calculus, what its operations are and on what sort of objects it operates. If we agree (without getting into arguments about definitions) that differentiation and integration are about *functions* defined on certain *sets of real numbers*, the problems are already simple to state: just what *are* the ‘real numbers’ and what should we demand of a ‘function’ before we can differentiate it? We may decide that the numbers should ‘lie on the number line’ and that the functions concerned should at least ‘tend to a limit’ when the points we are looking at (on the x-axis, say) ‘approach’ a particular point a on that line. And we may require even more.

Our basic problem is that these ideas involve *infinite sets* of numbers in a very fundamental way. We can represent the idea of *convergence*, for example, as a two-person game which in principle has infinitely many stages. In its simplest form, when we want to say that an infinite sequence (a_n) of real numbers has limit a, we have two players: player 1 provides an estimate of closeness, that is, she insists that the *distance* between a_n and a should be small enough; while player 2 then has to come up with a ‘stage’ in the sequence beyond which *all* the a_n will satisfy player l’s requirements. To make this more precise: if player 1 nominates an ‘error bound’ $\varepsilon > 0$ then player 2 has to find a positive integer N such that for *all greater* integers n the distance between a_n and a is less than ε. Only then has player 2 successfully survived that phase of the game. But now player 1 has infinitely many further attempts available: she can now nominate a different error bound ε and player 2 again needs to find a suitable N, over and over again. So player 2 wins if he can *always* find a suitable N, whatever choices of ε player 1 stipulates; otherwise player 1 will win.

This imaginary game captures the spirit of convergence: however small the given error bound ε, it must always be possible to satisfy it from some point onwards. This idea cannot easily be translated into something a computer can check! Computers can check case after case, very quickly, and this can provide useful information, but they cannot provide a *proof* that the conditions will always be met.

Another simple example of this comes when we want to show that there are infinitely many prime numbers. The computer cannot check them all, precisely because there are infinitely many. However, a *proof* of this fact has been known for thousands of years, and is recorded, for example, in the Greek mathematician Euclid's famous *Elements of Geometry*, written about 300 BC. The idea is simple enough: if there were only finitely many, there would be a largest prime, p, say. But then it is not hard to show that (using the factorial $p! = 1.2.3 \dots (p - 1).p$) the number $K = p! + 1$ is also prime, and is bigger than p. Thus p can't be the largest prime, and so the claim that such a prime exists leads us to a contradiction. Hence there must be infinitely many primes.

Here, as so often in mathematical proofs, the *logical sequence* of our statements is crucial, and enables us to make assertions about infinite sets, even though we are quite unable to verify each possible case separately in turn. While computers can be taught the latter, the *analytical skills* and the ability to handle *abstract concepts* inherent in such reasoning still have the dominant role in mathematics today.

This is also the stated aim of mathematics at A-level in the UK: one of the 'compulsory assessment objectives' in A-level Mathematics states that students should be able to: *construct a proof or a mathematical argument through an appropriate use of precise statements, logical deduction and inference and by the manipulation of mathematical expressions*. This book is not a political tract, so we shall refrain from commenting on how far this aim is achieved in practice.

This book is written for university students who, by and large, will have had some exposure to mathematical proof and logical deduction. Our main aim is to provide a body of closely argued material on which these skills can be honed, in readiness for the higher levels of abstraction that will follow in later years of your undergraduate course. This process requires patience, effort and perseverance, but the skills you should gain will pay off handsomely in the end.

1.2 From counting to complex numbers

First of all, what *is* a real number? We need to decide this; otherwise we can't hope to talk about *sequences* of real numbers, *functions* which take real numbers to real numbers, etc. The pictorial representation as all numbers on an infinite line is a useful guide, but hardly an adequate *definition*. In fact, through the centuries there have been many views of what this *continuum* really represents: is a line just a 'collection of dots' spaced infinitely closely together, or is it an indivisible whole, so that, however small the pieces into which we cut it, each piece is again a 'little line'? These competing points of view lead to very different perceptions of mathematics.

But let us start further back. What do we need sets of numbers for? One plausible view is that we can start with the set N of natural numbers as given; in the nineteenth century German mathematician Leopold Kronecker's famous phrase: *God created the natural numbers; all the rest is the work of Man*. We are quickly led to the set Z of all (positive and negative) integers, since positive integers allow addition, but not (always) subtraction. Within Z we can multiply numbers happily, but we cannot always divide them by each other (except by 1). Thus we consider ratios of integers, and create the set Q of all rational numbers, in which all four operations of arithmetic are possible, and keep us within the set. However, now we find – as the followers of the Greek philosopher and mathematician Pythagoras discovered to their evident dismay around 450 BC – that *square roots* present a problem: the set Q is found to have 'gaps' in it, for example where the 'number', $\sqrt{2}$ should be. Plugging this gap took rather a long time, and led to many detours on the way: a modern description of this journey takes up Chapters 2–5 of the companion text *Numbers, Sequences and Series* by Keith Hirst. (Historically, the *axiomatic* approach which is now taken to these problems is a recent phenomenon, even though the Ancient Greeks introduced the axiomatic method into geometry well over 2000 years ago.)

And even then mathematicians were not satisfied, since, although the equation $x^2 - 2 = 0$ could now be solved, and its solutions, $x = \sqrt{2}$, and $x = -\sqrt{2}$ made sense as members of the set R of real numbers, the solutions of $x^2 + 1 = 0$ did not! The final step, to the system C of complex numbers, occupies Chapter 6 of Hirst's book – here, however, we shall stick to the set R of real numbers for our Analysis.

1.3 From infinitesimals to limits

Our main concern is not with the properties of the set R as a single entity, but rather with the way in which its elements relate to each other. Thus we shall take the algebraic and order properties of R for granted, and focus on the consequences of the claim that R 'has no gaps'.

Just what this means bothered the Greek mathematicians considerably. The idea that lines and curves are 'made up' of dots, or even of 'infinitely short lines' is quite appealing: it allowed mathematicians to imagine that, by adding points 'one by one' to a line segment they could measure *infinitesimal* increases or decreases in its length. The known properties of regular rectilinear bodies could then be transferred to more complicated curvilinear ones. A circle, for example, could be imagined as a polygon with infinitely many infinitely short sides: which leads to a simple proof of the area formula: imagine the circle of radius r as made up of infinitely many infinitely thin isosceles triangles, each with height infinitely close to r and infinitesimal base b. Each has area equal $\frac{1}{2}br$, so the area of the circle is $A = \frac{1}{2}rC$, where C (the sum of all the bases b) is the circumference of the circle. But if π is the ratio of circumference to diameter, we also have $C = 2\pi r$, so that, substituting for C, we obtain $A = \pi r^2$.

Though the logical difficulties of adding infinitely many quantities (while their sum remained finite) and dividing finite quantities by infinitesimals soon discredited such techniques, they have stayed with mathematicians throughout the centuries as useful heuristic devices. They flourished again in build-up to the Calculus in the sixteenth and seventeenth centuries: Johann Kepler, for example, gave a three-dimensional version of the above argument, showing how the sphere is 'made up' of infinitely thin cones and hence that its volume V is $\frac{r}{3}$ times the surface area ($A = 4\pi r^2$), yielding $V = \frac{4}{3}\pi r^3$. Gottfried Wilhelm Leibniz, in particular, sought to put the infinitesimals dx on a proper logical footing in order to justify statements like

$$\frac{(x+\mathrm{d}x)^2 - x^2}{\mathrm{d}x} = 2x + \mathrm{d}x \approx 2x$$

where the symbol ≈ denotes that the quantities differ only by an infinitesimal amount. Much effort was expended to resolve the paradox involved in first dividing by the quantity *dx* and then ignoring it as if it were 0, and throughout the eighteenth and early nineteenth century this led to a gradual realization of how we could describe these ideas using *limits*, and that a proper analysis of the classes of *functions* which describe the curves involved should precede any justification of the Calculus. This led away from pictorial representation, and a closer look at the *number systems* on which such functions had to be defined. The wheel had now turned full circle, and by the early nineteenth century mathematicians could begin their study of the newly independent subject of Analysis.

2

Convergent Sequences and Series

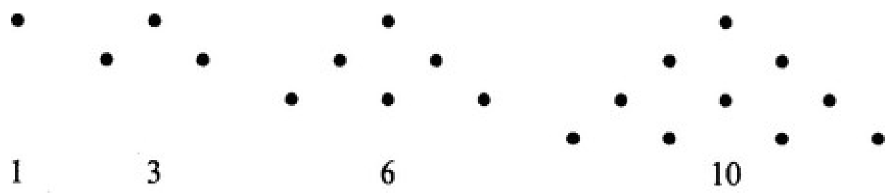
This chapter is devoted to a self-contained *review* of the properties of convergent sequences and series, which are described in more detail in the companion text *Numbers, Sequences and Series* by Keith Hirst, Chapters 7–9. This text will henceforth be referred to as [*NSS*]. If you have no previous experience of the fundamental idea of *convergence of a real sequence*, or wish to refresh your memory, you should consult this text and practise your skills on the examples and exercises provided there, which complement those presented in this book. The idea of convergence is a fundamental theme of the present book, and the results discussed in this chapter will be used throughout those that follow. The definitions of the number systems N, Z, Q, R and C will be taken for granted in this book: details of these can also be found in [*NSS*].

The terms 'sequence' and 'series' are often used interchangeably in ordinary language. This is a pity, since the distinction between them is very simple, and yet very useful and important. We shall take care *not* to confuse them.

2.1 Convergence and summation

Sequences are 'lists' of numbers, often generated by an inductive procedure, such as those familiar from early number games in which we have to 'guess' the next number. For example, given 1,3,6,10,15,… we might decide that the next number should be 21, since the difference between successive numbers increases by 1 each time. We either need to be given sufficiently many terms to deduce the rule of succession, or we can be given the rule directly.

Example 1: The above sequence of *triangular numbers* is obtained by writing $a_1 = 1$, $a_2 = 1 + 2 = a_1 + 2$, $a_3 = a_2 + 3$, etc. The reason for the name becomes clear when we represent each unit by a dot in the following pattern:



The properties of this sequence were well known to the Ancient Greeks. Pythagoras and his followers studied them in some detail, for example. For each $n \geq 1$, $a_n - a_n = n$. Therefore we can find the value of a_n by starting at $a_1 = 1$ and successively adding in all the differences, i.e. $a_n = 1 + 2 + 3 + \ldots + n$. The sum is easy to calculate:

$$a_n = 1 + 2 + \ldots + (n-1) + n$$
$$a_n = n + (n-1) + \ldots + 2 + 1$$

Adding, we have $2a_n = n(n+1)$, so we have proved the well-known formula

$$1 + 2 + 3 \ldots + n = \tfrac{1}{2}n(n+1)$$

In this example the inductive definition reduces to a *formula* which allows us to express a_n as a *function* of n, namely $a_n = \frac{1}{2}n(n+1)$. There are many such examples.

Pythagoras (*ca*. 580-495 BC) and his followers were perhaps the first group to attempt an explanation of their physical environment in quantitative terms, i.e. associating numbers with objects. Whole numbers served as the building blocks of the universe for them, and they were thus led into a study of the relationship between numbers, based in part on figurative numbers such as those shown above.

Example 2: The sequence {1, 4, 9, 16, …}, all consisting of perfect squares, is fully described by writing $\{n^2 : n \geq 1\}$; similarly we have two descriptions of the sequence $\{2^{-n} : n \geq 1\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$. In each case $a_n = f(n)$ for some function f.

In these examples the sequences generated are (at least potentially) *infinite:* they do not stop, and for each $n \in \mathbb{N}$ we can find the n^{th} *term* a_n of the sequence (a_n). In our examples these terms are also real numbers; hence a *real sequence* $(a_n)n \geq 1$ is thought of informally as an infinite list of real numbers, usually given by some rule for forming each term in the list.

All *infinite* sequences have one thing in common: to each $n \in \mathbb{N}$ there corresponds a uniquely defined real number a_n. We turn this into a quite general *definition:*

A **real sequence** is a function $n \mapsto a_n$ with domain N and with range contained in R.

Of course, we have not formally defined the term ‘function’ so far, and without this the above definition is not very meaningful. We shall give formal definitions in Chapter 5; for the present the simple idea of a function as a ‘rule’ suffices to make clear that we wish to associate a real number a_n with $n \in \mathbb{N}$

However, the *terms* a_m and a_n need not be different if $m \neq n$*:* for example, if $a_n = (-1)^n$, then every even-numbered term is 1 and each odd-numbered term is −1. Here the range of the function $n \mapsto a_n$ is just the two-point set { −1, 1}. Similarly, we can define *constant* sequences simply by setting $a_n = c$ (where c is a fixed real number) for all $n \in \mathbb{N}$.

The principal purpose of our analysis of infinite sequences is to provide a convenient tool for *approximation* and *convergence*. For example, we can approach the irrational number $\sqrt{2}$ by writing down successive decimal approximations, such as 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, 1.4142136, etc. Since $\sqrt{2}$ cannot be written as a finite decimal, the approximating sequence will never ‘reach’ the target value, $\sqrt{2}$ but by taking enough terms we can approximate it to as high a degree of accuracy as we wish.

Despite our formal definition we usually think of the terms of sequence (a_n) as values which are 'taken in succession' as n increases. This is especially useful in dealing with sequences which are defined *recursively:* if, for each x_n, its *successor* (the next term) x_{n+1} is defined as some function of x_n (and possibly some of its earlier *predecessors*, $x_{n-1}, x_{n-2}, \ldots, x_1$) then we may be able to guess what happens to x_n 'in the long run'.

Example 3: Set $x_1 = 1$, and for each n, suppose that $x_{n+1} = \frac{x_n^2 + 3}{2x_n}$. Here it is no longer clear how we might write $x_n = f(n)$ for some explicit function f, but we have $x_2 = 4/2 = 2$, $x_3 = 7/4 = 1.75$, and similarly $x_4 = 1.732143$, $x_5 = 1.732051$, etc. So it seems that this sequence 'settles down' rather quickly at $\sqrt{3}$. We can 'check' this by writing the 'limit' of the sequence as x, so that x *should* be a solution of the equation $x = \frac{x^2 + 3}{2x}$ (why?), and this obviously simplifies to $x^2 = 3$.

In fact, even the attempt to *define* irrationals like $\sqrt{3}$ via their decimal representation requires us to specify precisely what we mean by the 'limiting value' of a sequence: we could consider the sequence $(x_n)_{n \geq 1}$ whose terms are given successively as $x_1 = 1$, $x_2 = 1.7$, $x_3 = 1.73$, $x_4 = 1.732$, etc. — and then the requirement is to make sense of 'x_∞' (!) We shall solve this problem shortly.

TUTORIAL PROBLEM 2.1

The use of *iteration* to provide approximate solutions to equations can be taken much further: read [*NSS*] Chapter 7.2 for more examples and test your skills on the Exercises at the end of that section. We shall return to these problems in Chapter 7, when we will be able to justify the techniques used much more fully.

Series are, informally, 'what you get when you add together the terms of a sequence'. But we cannot add infinitely many numbers together directly, so this vague claim needs to be made more precise. What we do, given the terms of a sequence $(a_i)_{i \geq 1}$, is to form the *partial sums* $s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3$, etc., and in general:

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i$$

For each $n \geq 1$ we call s_n the n^{th} partial sum of the *series* $\sum_i a_i$. Strictly speaking, the series is itself defined by this sequence of partial sums, so that a formal definition of a series involves listing the sequence of its terms (a_i) as well as the sequence (s_n) of partial sums — and each can be found from the other (*see [NSS]*, Definition 8.1 for a more formal description).

We shall reserve the notation $\sum_i a_i$ for the series whose terms are given by members of the sequence (a_i).

Of course, this still doesn't answer the question of what we should mean by the *sum* of the series $\sum_i a_i$. Fortunately, we can decide this as soon as we know how to define the 'limiting value' 's_∞.' of the sequence of its partial sums, so that we handle both our problems at once.

TUTORIAL PROBLEM 2.2

As an alternative to the usual 'Achilles and the Tortoise' example (*see [NSS]*, Chapter 8), imagine a snail crawling along infinitely expandable rubber bands in various ways. In each case we are given that the snail crawls 1 m each day, and that it rests at night. While it rests, a demon secretly stretches the rubber band. In each of the following examples the question is: does the snail (which starts from one end of the band) ever reach the other end, and, if so, how long does it take?

(i) The initial length of the rubber band is 10 m. Every night the demon stretches it by a further 10 m.

(ii) The initial length of the rubber band is 4 m. Every night the demon doubles its previous length.

(iii) his time the rubber band is initially 2 m long, and the demon doubles the length every night.

(iv) The rubber band is x metres long at the start, and the demon increases its length by 25% every night. What is the 'critical' value x^* so that for all $x < x^*$ the snail will reach the other end in a finite time? Given that $x^* > 4$, find how long the snail will take when $x = 4$.

Limits Of Infinite Sequences

The above examples all have one thing in common: we are interested in what happens 'eventually' or 'in the long run' as we move down our sequence (a_n) — or along the corresponding sequence of partial sums. If we want the sequence to 'have a limit x', then its terms will 'eventually' have to be 'as close as we please' to x. In other words, the distance between x_n and x becomes *arbitrarily small* provided we take n *sufficiently large*. This leads to the following fundamental concept:

Definition 1:

(i) The real sequence (x_n) *converges to the finite limit x as n tends to infinity* (written $x_n \to x$ as $n \to \infty$, or $x = \lim_{n \to \infty} x_n$) if: for every real $\varepsilon > 0$ there exists an

$N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n > N$.

(ii) If (x_n) has no finite limit, we say that the sequence *diverges*.

Formally, part (ii) includes the case when (x_n) 'goes to infinity': for example, if for every given $K > 0$ we can find $N \in \mathbb{N}$ such that $x_n > K$ whenever $n > N$.

By a slight abuse of notation we shall nevertheless write $x_n \to \infty$ when this happens. A similar interpretation is given to the notation $x_n \to -\infty$. Rather confusingly, some authors describe this by saying that '(x_n) diverges to ∞' (or − ∞).

In order to apply this definition to some useful examples we need just one property of the real line R at this stage — that R contains 'arbitrarily large' natural numbers. We formulate this, without proof, as the:

Principle of Archimedes

Given any real number x there exists $n \in \mathbb{N}$ such that $n > x$.

With a precise definition of R via a set of *axioms* one can prove this (rather obvious?) Principle rigorously – we shall show later that it follows easily from the *Completeness Axiom* for R. These issues are discussed fairly fully in [*NSS*], Chapter 5 – *see* Proposition 5.6. We shall explore the consequences of the Completeness Axiom for the convergence of sequences and series in more detail in Chapter 3.

Example 4: Let x_n = for for each $n \geq 1$. This is the sequence $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$ which should clearly have limit 0. To prove this, suppose some $\varepsilon > 0$ is given. How do we find an 'appropriate' N? Now if $n > N$ then $\frac{1}{n} < \frac{1}{N}$, hence we only need to find a *single* N such that $\frac{1}{N} < \varepsilon$, which is the same as demanding that $N > \frac{1}{\varepsilon}$ But since $\frac{1}{\varepsilon}$ is a real number, the Principle of Archimedes ensures that such an N must exist in N. Therefore we have proved that $\frac{1}{n} \to 0$ when $n \to \infty$. Note that the N we found above need *not* be the first integer greater than $\frac{1}{\varepsilon}$;*any* such integer will do! However, in general the number N will depend on the given ε, and so to prove convergence we need to supply a 'rule' which chooses $N = N(\varepsilon)$, and which does so for *every* given $\varepsilon > 0$.

Example 5: $x_n = \frac{n}{n+1}$ is the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots$ and it seems clear that its limit should be 1. To *prove* this we need to consider the *distance* $|x_n - 1| = 1 - \frac{n}{n+1} = \frac{1}{n+1}$ and since this clearly converges to 0 as $n \to \infty$ it follows at once that $\lim_{n\to\infty} x_n = 1$.

Example 6: On the other hand, the sequence −1,1, −1,1, −1,1, −1,1, …, which we express conveniently as $x_n = (-1)^n$, seems never to settle down, and the difference between consecutive terms is always 2, since the values of the terms *oscillate* forever between −1 and 1.

To prove that (x_n) *diverges*, note first that $|x_n.- x_{n+1}| = 2$ for all n, as observed above. Now if the sequence (x_n) had a limit x, say, then $\lim_{n\to\infty} x_{n+1} = x$also, by the definition of convergence. So we would be able to find N' and N'' such that $|x_n - x| < 1$ for $n > N'$, and $|x.- x_{n+1}| < 1$ for $n > N''$. Hence for all $n > N = \max(N', N'')$ both these inequalities must hold. But this contradicts the fact that $|x_n.- x_{n+1}| = 2$.